Assignment 3 - Solutions

1. Consider a set S of C-programs. (i) Define three meaningful binary relations on S. (ii) Define a relation which is not transitive.

   R_1=$\{(a, b) \mid a$ and b have same asymptotic time complexity$\}$
   R_2=$\{(a, b) \mid a$ and b both are comparison based sorting$\}$
   R_3=$\{(a, b) \mid a$ and b both finds maximum element in the input$\}$

   R_4=$\{(a, b) \mid a$ and b have common functions$\}$

2. Consider a set S of academic courses offered at IIITDM. (i) Define three meaningful binary relations on S. (ii) Define a relation which is not transitive.

   R_1=$\{(a, b) \mid a$ is a prerequisite for $b\}$
   R_2=$\{(a, b) \mid a$ and b have both lab and theory$\}$
   R_3=$\{(a, b) \mid a$ and b are courses in same semester$\}$

   R_4=$\{(a, b) \mid a$ and b are related iff they are from different departments$\}$

3. Consider a set S of coins with different denominations. (i) Define three meaningful binary relations on S. (ii) Define a relation which is not transitive.

   R_1=$\{(a, b) \mid a + b = 100\}$
   R_2=$\{(a, b) \mid a \bmod b = 2\}$
   $R_3 = \{(a, b) | a \leq b\}$

   $R_4 = \{(a, b) | a - b = 10\}$

4. On the set of natural numbers, define a binary relation which is not transitive.

   $R_1 = \{(a, b) | a + b \leq 3\}$

5. Define a binary relation on empty set and list all properties that it satisfies.

   $A = \emptyset$
   $A \times A = \emptyset$
   $R = \emptyset$
   R is reflexive, symmetric, transitive, equivalence, irreflexive, antisymmetric, asymmetric and partial order.

6. Suppose, a binary relation R does not satisfy property P (reflexive, symmetric, etc.), then R can be made to satisfy P by adding or deleting elements (each element in R is a pair). Mention for each property, whether the operation is addition or deletion. Also, suppose R contains k elements, what is the minimum and maximum number of elements to be added/deleted to satisfy P. The underlying set has n elements.

Reflexive: Only addition is possible.
Add: Minimum $= n - k$ (all k elements are of the form (a_i, a_i), the remaining elements are $n - k$) , Maximum $= n$ (none of (a_i, a_i) exists).
Symmetric:
Minimum $= 1$ (element can be added / deleted assuming only one element (a, b) does not have (b, a)), Maximum $= k$ (none of (a_i, b_i) has (b_i, a_i) pair.)
Irreflexive:
Delete: Minimum=1 (only one pair (a_i, a_i) exists) , Maximum $= k$ (all k elements are of the form (a_i, a_i)).
Antisymmetric:
Delete: Minimum=1 (assume only one element has (a, b) and its symmetric pair (b, a)) , Maximum $= \frac{k}{2}$ (assume all the elements (a_i, b_i) has its symmetric pair (b_i, a_i))
Asymmetric:
Delete: Minimum= 1 (assume only one element has (a, b) and its symmetric pair (b, a) or one (a, a) is present) , if $k < n$ Maximum $= k$ (all elements (a_i, a_i) are present) and if $k > n$, Maximum $= n + \frac{(k-n)}{2}$
Transitive:
Add: Minimum=1 (one of (a, b) and (b, c) does not have (a, c)) , Maximum is attained when R contains all elements from first row and first column, then $|k| = 2n - 1$. To make R transitive we need to add all the remaining elements. Maximum $= n^2 - (2n - 1) = n^2 - k$.

7. Given a set A of size n and a relation $R \subseteq A \times A$, how many relations are (i) either reflexive or symmetric (ii) reflexive or symmetric (iii) Neither reflexive nor symmetric.
$n(R)$= Number of reflexive binary relation $= 2^{n^2-n}$
$n(S)$= Number of symmetric binary relation=$2^{(n^2+n)/2}$
$n(R \cap S)$=$2^{(n^2-n)/2}$

   (i) either reflexive or symmetric= $n(R)+n(S)-2\times n(R\cap S)$ =$2^{n^2-n}+2^{(n^2+n)/2}-2\times 2^{(n^2-n)/2}$
=$2^{n^2-n} + 2^{(n^2+n)/2} - 2^{(n^2-n+2)/2}$
(ii)reflexive or symmetric=$n(R) + n(S) - n(R \cap S)$
=$2^{n^2-n} + 2^{(n^2+n)/2} - 2^{(n^2-n)/2}$
$= 2^{n^2-n} + 2^{(n^2+n)/2} - 2^{(n^2-n)/2}$
(iii) Neither reflexive nor symmetric= 2^{n^2} -reflexive or symmetric.
$2^{n^2} - (2^{n^2-n} + 2^{(n^2+n)/2} - 2^{(n^2-n)/2})$

8. Given an integer k, how many reflexive binary relations are there containing exactly k elements. Assume, the set on which the relation is defined has n elements.
case i: $k < n$, no reflexive relation is possible
case ii: $k = n$, Only one reflexive relation is possible
case iii:$k > n$, among n^2 elements, n diagonal elements should be present and from the remaining $n^2 - n$ elements $k - n$ elements can be chosen.
$n^2 - n_{C_{k-n}}$ reflexive binary relations are possible.

9. Given an integer k, how many antisymmetric binary relations are there containing exactly k elements. Assume, the set on which the relation is defined has n elements.
All k elements can be choosen from the diagonal or it can be chosen from $n^2 - n$ or some from diagonal and remaining from $n^2 - n$. For $n^2 - n$ elements, there are two possibilities, either (a, b) or (b, a) can be present.

for $k \leq n$,
number of antisymmetric binary relations=$\sum_{i=0}^{k} n_{C_i} \times ((n^2-n)/2)_{C_{k-i}} \times 2^{k-i}$
for $k > n$,
number of antisymmetric binary relations=$\sum_{i=0}^{n} n_{C_i} \times ((n^2-n)/2)_{C_{k-i}} \times 2^{k-i}$
for $k = 0$, number of antisymmetric binary relations=1

10. Mention the smallest set A and the smallest relation R defined on A such that R does not satisfy any of the properties.
$A = \{1, 2, 3\}$
$R = \{(1, 1), (1, 2), (2, 1), (1, 3)\}$

11. Write the definition of rational number, irrational number and real number in FOL.
Rational
R is a rational number $\leftrightarrow \exists p \exists q[p, q \in \mathbb{I} \wedge q \neq 0 \wedge R = \frac{p}{q}]$
Irrational
R is a irrational number $\leftrightarrow \forall p \forall q[p, q \in \mathbb{I} \wedge q \neq 0 \wedge R \neq \frac{p}{q}]$
Real
R is a real number $\leftrightarrow \exists p \exists q[p, q \in \mathbb{I} \wedge q \neq 0 \wedge R = \frac{p}{q}] \oplus \forall p \forall q[p, q \in \mathbb{I} \wedge q \neq 0 \wedge (R \neq \frac{p}{q})]$

12. Prove using the principle of mathematical induction the number of antisymmetric binary relations on a set of size n.
Number of antisymmetric relations(say x)=$2^n.3^{n(n-1)/2}$
By induction on n:
Base case: $n = 0,\ A = \emptyset, R = \emptyset, x = 1$
$n = 1, A = \{1\}, R_1 = \{(1, 1)\}, R_2 = \emptyset$
Induction Hypothesis: Assume that the number of antisymmetric relations on a set of size k is $2^k.3^{k(k-1)/2}$ where k$\geq$1
Induction Step:
we have to prove that antisymmetric relation on set of size $(k + 1)$ is $2^{(k+1)}.3^{(k+1)((k+1)-1)/2}$.
Antisymmetric relation on set of size $(k+1)$=Antisymmetric relation on set of size k + Newly created antisymmetric relation because of adding $(k + 1)$.
Antisymmetric relation on set of size k = $2^k.3^{k(k-1)/2}$ (By Induction Hypothesis).
Antisymmetric relation because of adding $(k+1)$= Newly added pairs, for example, $(1, k+1)$ or $(k + 1, 1)$ or none of the both are present. This is true for $(1, k + 1)$,$(2, k + 1)$ and so on upto $(k, k + 1)$. The pair $(k + 1, k + 1)$ may or may not be present.
Each of terms newly added because of $k + 1$ has three choices either include one element or other one or not both. $(k + 1, k + 1)$ has two choices.
Antisymmetric relation because of adding $(k + 1)$=$3^k \cdot 2$.
Antisymmetric relation on set of size $(k + 1)$=$2^k.3^{k(k-1)/2}.3^k.2$
=$2^{(k+1)}.3^{(k+1)((k+1)-1)/2}$
=$2^{(k+1)}.3^{(k+1)(k)/2}$

13. Prove using MI: $x^0 = 1$.
Base case: $x = 1, 1^0 = 1$.
$1^{1-1} = 1/1 = 1$ is true.
Induction Hypothesis: Assume $k^0 = 1$ is true for $k \geq 1$.
Induction step: To prove for $(k + 1)$
$(k + 1)^0 = 1$(by binomial expansion)
$(k + 1)^0 = 0!/(0!0!) \cdot k^0 \cdot 1^0$ ($k^0 = 1$ by hypothesis and $1^0 = 1$ by base case)

$(k+1)^0 = 1$.
Hence Proved.
For Negative Integers, Mathematical Induction cannot be applied.
Justification:
Set of $\mathbb{I}^+$ is a well ordered set. There exists a least element which is the base case for M.I. But a Set of $\mathbb{I}^-$ is not well ordered since it has no least element. Hence there does not exist a base case to apply M.I.
If we redefine the notion 'successor of an element', then MI can be applied on $\mathbb{I}^-$ as well. Base: $k = -1$, Hypothesis: Assume $k^0 = 1$, for $k \leq -1$. Induction step: The successor of k is now $k-1$, not $k+1$. $(k-1)^0 = 0!/(0!0!) \cdot k^0 \cdot (-1)^0$ ($k^0 = 1$ by hypothesis and $(-1)^0 = 1$ by base case)

$(k+1)^0 = 1$.
Hence Proved.

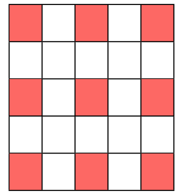
Method 2:
Base case: $x = 1,\ 1^0 = 1^{1-1} = \frac{1}{1} = 1 \implies 1 = 1$.
Induction Hypothesis: Assume n^0 is true for $n \geq 1$
$n^0 = n^{1-1} = n^1 \times n^{-1} = \frac{n^1}{n^1} = \frac{n}{n} = 1 \implies n = n$
Induction Step: prove $(n+1)^0 = 1$
from hypothesis and base case,
$1 = 1$ —>A
$n = n$ —>B
A+B $\implies n + 1 = n + 1$
$\frac{n+1}{n+1} = 1$
$(n+1)^{1-1} = 1$
$(n+1)^0 = 1$
$\therefore x^0 = 1$

Method 3:
Let 'n' be any other number other than '0'
we know that, $n^a \cdot n^b = n^{a+b}$
$n^x \cdot n^0 = n^{x+0} = n^x$ $\quad --- >$A
$n^x \cdot 1 = n^x$(Any number multiplied with one is equal to the same number) $\quad --- >$B
we know that $a = b$ and $c = b \implies a = c$
from A and B: $a = n^x \cdot n^0$, $b = n^x$ and $c = n^x \cdot 1$
$n^x \cdot n^0 = n^x \cdot 1$
$\therefore n^0 = 1$

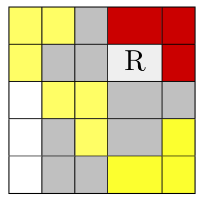
similarly for negative numbers,
we know that, $(-n)^a \cdot (-n)^b = (-n)^{a+b}$
$(-n)^x \cdot (-n)^0 = (-n)^{x+0} = (-n)^x$ $\quad --- >$A
$(-n)^x \cdot 1 = (-n)^x$ (Any number multiplied with one is equal to the same number) $\quad --- >$B
we know that $a = b$ and $c = b \implies a = c$
from A and B: $a = (-n)^x \cdot (-n)^0$, $b = (-n)^x$ and $c = (-n)^x \cdot 1$
$(-n)^x \cdot (-n)^0 = (-n)^x \cot 1$
$\therefore (-n)^0 = 1$

14. The set A consists of all reflexive binary relations and B consists of all antisymmetric binary relations.Between the two, which set is larger. Justify.
Number of reflexive binary relations=2^{n^2-n}
Number of antisymmetric binary relations=$2^n.3^{(n^2-n)/2}$
$2^{n^2-n} > 2^n.3^{\frac{(n^2-n)}{2}}$
proof by M.I:
Base case:
$n = 6$
$2^{30} > 2^6 \cdot 3^{15}$
16777216>14348907 is true.
Induction Hypothesis: Assume $2^{k^2-k} > 2^k.3^{\frac{(k^2-k)}{2}}$ is true for $k \geq 6$.
Induction step: To prove $2^{(k+1)^2-(k+1)} > 2^{(k+1)}.3^{\frac{(k+1)^2-(k+1)}{2}}$.
$=2^{(k+1)}.3^{\frac{(k+1)^2-(k+1)}{2}}$.
$=2^k \cdot 2 \cdot 3^{\frac{k^2+2k+1-k-1}{2}}$
$=2^k \cdot 2 \cdot 3^{\frac{k^2+k}{2}}$
$=2^k \cdot 2 \cdot 3^{\frac{k^2-k}{2}+k}$
$=2^k \cdot 2 \cdot 3^{\frac{k^2-k}{2}} \cdot 3^k$
$=2^k \cdot 3^{\frac{k^2-k}{2}} \cdot 2 \cdot 3^k$
by our hypothesis
$< 2^{k^2-k} \cdot 2 \cdot 3^k$
since $3^k < \frac{4^k}{2}$ replace 3^k by $\frac{4^k}{2}$
$< 2^{k^2-k} \cdot 2 \cdot \frac{4^k}{2}$
$= 2^{k^2-k} \cdot 2^{2k}$
$= 2^{k^2-k+2k}$
$= 2^{k^2+k}$
hence it is proved.
Therefore,Number of reflexive binary relations is greater than Number of antisymmetric binary relations.
Note: proof of $3^k < \frac{4^k}{2}$ using M.I Base case: $n = 3$
$3^3 < \frac{4^3}{2}$
$27 < 32$ is true
Induction Hypothesis: Assume $3^k < \frac{4^k}{2}$ is true for $k \geq 3$.
Induction step $3^{(k+1)} < \frac{4^{(k+1)}}{2}$
$=\frac{4^{(k+1)}}{2}$
$=\frac{4^{(k)} \cdot 4^1)}{2}$
$=\frac{4^k}{2} \cdot 4$
$> 3^k \cdot 4$
$> 3^k \cdot 3$
$= 3^{(k+1)}$
$3^{(k+1)} < \frac{4^{(k+1)}}{2}$
$\therefore 3^k < \frac{4^k}{2}$

15. Is it true that $5 \times 5 - 1$ chess board can be filled using L-shaped triominos. Prove your answer.



For $5 \times 5 - 1$ chess board, Except shaded regions if any of unshaded region is left out then it is not possible to place the triominos.



Consider the left out cell as R. Red shaded portion has only one possibility to place the triomino. From the remaining cells, the smallest portion that can be filled completely by using triomino is either 2×3 or 3×2. If we try to partition the remaining cells of chessboard then either 1×3 or 3×1 is left out which cannot be filled using triomino.

16. Show that $(n/e)^n \leq n! \leq n^n$
claim(i): $n! \geq \frac{n^n}{e^n}$
we know that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + \frac{x^{(n+1)}}{(n+1)!} + ...$
$e^x \geq \frac{x^n}{n!}$
$x = n$
$e^n \geq \frac{n^n}{n!}$
$n! \geq \frac{n^n}{e^n}$
claim(ii):n^n $\geq$n!
$n \times n \times \cdots n \times n \geq n \times n - 1 \times \cdots 2 \times 1$
clearly LHS $\geq$ RHS.
hence proved.

Method 2:

claim(i): $n! \geq \frac{n^n}{e^n}$
$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$
Let $x = n,\ e^n = \sum_{k=0}^{\infty} \frac{1}{k!} n^k$
Multiply both sides by $n!$,
$n! e^n = n! \sum_{k=0}^{\infty} \frac{1}{k!} n^k$
taking n^{th} and separating it in RHS,
$n! e^n = n^n + n! \sum_{k=0, k \neq n}^{\infty} \frac{1}{k!} n^k$
$n! e^n \geq n^n$
$n! \geq \frac{n^n}{e^n}$

$n! \geq (\frac{n}{e})^n$

claim(ii):$n! \leq n^n$
Arithmetic Mean $\geq$ Geometric Mean
$\frac{1+2+...+n}{n} \geq \sqrt[n]{n!}$
$(\frac{n(n+1)}{2n})^n \geq n!$
$(\frac{(n+1)}{2})^n \geq n!$
$n! \leq (n+1)^n$
Multiply $n+1$ on both sides
$n! \times (n+1) \leq (n+1)^n \times (n+1)$
$(n+1)! \leq (n+1)^{n+1}$
substitute $n+1 = k$
$k! \leq k^k$
substitute $k = n \implies n! \leq n^n$

Method 3:(By MI)

claim(i): $n^n \geq n!$
Base case: $n = 1,\ 1! \leq 1^1,\ 1 \leq 1$ which is true.
Induction Hypothesis: Assume $k! \leq k^k$ is true for $k \geq 1$
Induction Step: To prove $(k+1)! \leq (k+1)^{k+1}$
we already know that $k! \leq k^k$ by the induction hypothesis
also, for positive numbers, $k^k \leq (k+1)^k$
therefore, $k! \leq (k+1)^k$
Multiply $(k+1)$ both sides of the inequality,
$(k+1)k! \leq (k+1)^k(k+1)$
$(k+1)! \leq (k+1)^{k+1}$
$\therefore \forall n,\ n! \leq n^n$

claim(ii): $n! \geq (\frac{n}{e})^n$
Base case: $n = 1,\ \frac{1}{e} \leq 1$ which is true.
Induction Hypothesis: assume $k! \geq (\frac{k}{e})^k$ is true for $k \geq 1$.
Induction Step: To prove $(\frac{k+1}{e})^{(k+1)} \leq (k+1)!$
$(k+1)! = (k+1)k!$
from the hypothesis, $(k+1)k! \geq (k+1)(\frac{k}{e})^k$
Multiply and divide by $(k+1)^k$ in RHS
$(k+1)! \geq (k+1)\frac{k^k}{(k+1)^k}\frac{(k+1)^k}{e^k}$
dividing both sides by e,
$(\frac{k}{k+1})^k(\frac{k+1}{e})^{k+1} \leq \frac{(k+1)!}{e}$

By Properties of limits, $e > (1+\frac{1}{k})^k$
$\frac{1}{e} < \frac{1}{(1+\frac{1}{k})^k}$
$\frac{1}{e} < \frac{k^k}{(k+1)^k}$

Since, we have shown that $\frac{1}{e} < \frac{k^k}{(k+1)^k}$ and the above inequality is true, will only imply that $(\frac{k+1}{e})^{(k+1)} \leq (k+1)!$ has to be true.

$\therefore (\frac{n}{e})^n \leq n!$

17. Present a good lower bound and an upper bound for the number of transitive binary relations.
    Any subset of the set of diagonal elements is an example of transitive relation and thus there are at least 2^n transitive relations.
    lower bound:2^n
    Total number of relations minus non transitive relations gives the upper bound.
    Irreflexive symmetric (except empty relation) is non transitive.
    upper bound:$2^{n^2} - 2^{(n^2-n)/2} + 1$

18. Given Rupee 5 and 7 denominations, show that any positive integer $x \geq k$, change for x can be given using these two denominations, where k is the base value.
    Base case: for $x = 24$, two $Rs.5$ and two $Rs.7$ are required to make $Rs.24$
    Induction Hypothesis: Assume for $x \geq 24$, there exists a change for x using denominations $Rs5$ and $Rs7$.
    Induction Step: we have to prove that it is possible to make up exactly a change for $Rs.k+1$ using $Rs.5$ and $Rs.7$.
    case (i): suppose there exists change for $Rs.k$ using at least two $Rs.7$. Replacing two $Rs.7$ by three $Rs.5$ will yield a change of $Rs.k+1$.
    case (ii): suppose there exists a change for $Rs.k$ using at least four $Rs.5$. Replacing four $Rs.5$ by three $Rs.7$ will yield change of $Rs.k+1$.
    Therefore, the claim follows.

19. For a ternary relation, how do you define reflexivity, symmetricity and transitivity. Present a formal definition using FOL.
    Reflexivity :$\forall x \in A(x,x,x) \in R$
    Symmetricity:$\forall x,y,z \in A((x,y,z) \in R \rightarrow (x,z,y) \in R \wedge (y,x,z) \in R \wedge (y,z,x) \in R \wedge (z,x,y) \in R \wedge (z,y,x) \in R)$
    Transitivity:$\forall x,y,z,w,v \in A((x,y,z) \in R \wedge (y,z,w) \in R \wedge (z,w,v) \in R \leftrightarrow (x,w,v) \in R)$.

20. This is for you. Ask an interesting question based on the material covered on relations. Also, present a solution to the question posed.

    Count the number of ways to partition a set of n objects into k non-empty subsets.
    Stirling Number of the Second Kind. The number of ways of partitioning a set of elements into non-empty sets is also called as the Stirling set number.
    Let $s(n,k)$ denote the number of ways of partitioning a n-sized set into k non-empty subsets.
    This can be counted using
    $s(n,k) = k \cdot s(n-1,k) + s(n-1,k-1)$
    when we add n^{th} element to create k partitions, there are two possibilities:

    (a) It is added as a single element set (singleton set) to existing $(k-1)$ partitions on $(n-1)$ elements so that we get k partitions on n elements, i.e., $s(n-1,k-1)$
    (b) It is added as an element to a set in k partitions on $(n-1)$ elements, i.e., there are k possibilities and hence, $k \cdot s(n-1,k)$

For example, The number of $k = 2$ partitions on $\{1, 2, 3\}$ can be obtained as follows;
(a) Fixing '3' as n^{th} element, the number of one sized partition on $\{1, 2\}$ is one; $\{\{1, 2\}\}$, Therefore, the number of one sized partition on $n-1 = 2$ elements is $s(n-1 = 2, k-1 = 1) = 1$. If n^{th} element is included in a separate partition, then we get $\{\{1, 2\}, \{3\}\}$, thus we get, $s(n = 3, k = 2) = 1$.
(b) The number of two sized partitions on two elements out of $\{1, 2, 3\}$; $\{\{1\}, \{2\}\}$ (for element '3'), $s(n - 1 = 2, k = 2) = 1$
To obtain $k = 2$ partitions on $\{1, 2, 3\}$, we add the third element to each partition; $\{\{1, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}\}$. Also, note that we do not need consider the argument by fixing the elements '1' and '2' as it is implicitly taken care in the above argument itself.

Therefore, $s(n, k) = k \cdot s(n - 1, k) + s(n - 1, k - 1)$
Observation:
The above problem finds the number of ways to partition a set of n objects into exactly k non-empty subsets.
Bell's number computes the number of ways of partitioning a set of n objects into all possible sized subsets. Thus, we can compute Bell's number as a function of $s(n, k)$.
$B_n = \sum_{k=1}^{k=n} s(n, k)$.