Group Assignment 4, Nov 04.

1. Solve GATE questions on relations, functions and proof techniques.

2. In how many different ways 121 students can be mapped to 6 different grades. Present two different solutions.
   (i) Each student has 6 possible grades, therefore, $6 \times 6 \times \ldots \times 6$ (121 times)$=6^{121}$.

   (ii) The number of functions from 121 sized domain to 6 sized codomain $= 6^{121}$.

3. Prove using MI; the number of bijective functions on a set of size n is $n!$.
   `Base:` $n = 1$, there exists exactly one function which is bijection as well. `Hypothesis:` Assume the claim is true for $n \geq 1$. `Induction:` For the $(n+1)^{th}$ element in the domain, there are $(n+1)$ choices in the codomain. Consider a function obtained from the induction hypothesis; if suppose $f(a_i) = a_j$, $1 \leq i \leq n$, $1 \leq j \leq n$, then one can obtain a corresponding function in the induction step as follows; (i) $f(a_i) = a_{n+1}$ and $f(a_{n+1}) = a_j$ or (ii) $f(a_{n+1}) = a_{n+1}$. This shows that, from each of the function given by the induction hypothesis, one can obtain $(n+1)$ more bijective functions. Thus, $n!(n+1) = (n+1)!$.

4. Prove using MI; the number of one-one functions is n_{P_m}.
   Induction on m by fixing n. Indn step: Consider a domain of size $m+1$. To each of the function obtained from the induction hypothesis, the $(m+1)^{th}$ element can be mapped to any one of $(n-m)$ elements in the codomain. Thus, the number of one-one functions is $n_{P_m}(n-m) = \dfrac{n!}{(n-m-1)!(n-m)}(n-m) = \dfrac{n!}{n-(m+1)} = n_{P_{m+1}}$

5. How many different onto functions are there from a set of size 5 to 3.
   $3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5$

6. What is wrong with the following proof; counting onto function: The number of onto functions from a set of size n to a set of size m $(n \geq m)$ is equivalent to distributing n pigeons into m pigeon holes such that each pigeon hole is non-empty. This number is precisely, the number of integral solutions to $x_1 + x_2 + \ldots + x_m = n$, $x_i \geq 1$.
   PHP assumes all pigeons are indistinguishable, whereas the elements in domain are distinguishable. Therefore, the above counting is undercounting.

7. (Courtesy: Wajahathullah) The number of onto functions is $n_{P_m}(m^{n-m})$. That is, select m elements from the domain and for the rest of the elements $(n-m)$, count all possible functions from a set of size $n-m$ to a set of size m. What is wrong with this counting of onto functions.
   The above counting is overcounting. There is a overlap between n_{P_m} and (m^{n-m}). For example $A = \{1, 2, 3\}, B = \{a, b\}$. The function $f(1) = a, f(2) = b$, assume $f(3) = a$. The same function is counted again, $g(3) = a, g(2) = b$, assume $g(1) = a$.

8. How many functions are there from A to A such that $f(A) \subset A$.
   Assume A is finite. The functions that satisfy the above property are precisely the class of non-onto functions. Since range is not equal to codomain, some elements in codomain do not have preimages. If A is infinite, then uncountable.

9. The number of functions from n to m $(n \geq m)$ such that each element in codomain has at least 2 pre-images.

total number of functions - nononto - onto functions with exactly one preimage for some element in codomain.
onto functions with exactly one preimage for some element in codomain:
assuming domain size m , codomain is n and $m \geq 2n$. Let $S(m,n)$ refers to onto functions such that each element in the codomain has at least 2 preimages. We shall use PIE; how many onto functions exactly one element in codomain has exactly one preimage, number of onto functions in which exactly two in the codomain have exactly one preimage, and so on.
$S(m,n) = 1!\binom{m}{1}\binom{n}{1}S(m-1,n-1) - 2!\binom{m}{2}\binom{n}{2}S(m-2,n-2) + \dots (-1)^{n-1}(n-1)!\binom{n}{n-1}\binom{m}{n-1}S(m-(n-1),n-1)$.

Solution 2: Counting functions such that each element in codomain has at least 2 pre-images is equivalent to counting 'partitions of size n such that each part is of size at least two'.
Let $T(m,n,\geq 2)$ mean partitions of size n on m elements such that each part is of size at least two.
Let $T(m,n,=1)$ mean partitions of size n on m elements such that exactly one part is of size exactly one.
Therefore;
$T(m,n,\geq 2) = n \cdot T(m-1,n,\geq 2) + T(m-1,n,=1)$; To n partitions of size $(m-1)$, m^{th} element can be added to any parts, therefore n ways. Further, if a part has size exactly one, then m^{th} element can be added to that as well.

$T(m-1,n,=1) = T(m-2,n-1,\geq 2)$ Consider $(n-1)$ partitions on $(m-2)$ elements such that each part is of size at least two, add $(m-1)^{th}$ element to a new part, to obtain n partitions containing exactly one part of size one.

$m < 2n$, the no function is possible. Base case: $T(2n,n,\geq 2) = \binom{2n}{2}\binom{2n-2}{2}...\binom{4}{2}$.

10. Consider a three degree polynomial $a_3x^3 + a_2x^2 + a_1x^1 + a_0x^0$ with rational co-efficients; how many different three degree polynomials are possible?
    Each $a_i \in Q$ and the possible values for coefficients are $(Q \times Q \times Q \times Q)$. This set is countably infinite.

11. Using PIE, how many bit strings (binary) of length eight do not contain six consecutive 0's.
    Number of bit strings of length 8 do not contain six consecutive 0's = Total number of bit strings of length 8 - Number of bit strings containing 6 consecutive 0's.
    Total number of bit strings of length 8 = $2^8 = 256$
    Let k denote the substring with six zeroes and a, b be a bit.
    Number of bit strings containing 6 consecutive 0's = number of bit strings of length 8 of the form kab + number of bit strings of length 8 of the form abk + number of bit strings of length 8 of the form akb - number of bit strings of length 8 of the form kab and of the form abk - number of bit strings of length 8 of the form abk and of the form akb - number of bit strings of length 8 of the form kab and of the form akb + number of bit strings of length 8 of the form kab, abk and akb.
    $= 4 + 4 + 4 - 1 - 2 - 2 + 1 = 8$
    Therefore, the number of bit strings of length 8 do not contain six consecutive 0's = 256-8 = 248.

12. Compare the following sets; All are uncountable and of same cardinality. One can establish a bijection from each set to R.
    (a) $A = [2,6], B = [0,1]$
    (b) $A = [0,1], B = R$
    (c) $A = (0,1), B = [0,1]$

Solution:
(i) $A = [2, 6]$, $B = [0, 1]$
Solution 1:
$f(x) = \frac{(x-2)}{4}$ $\forall x \in [2, 6]$
Range of $f(x) = [0, 1] = B$. Therefore, it is an onto function. We shall now show that $f(x)$ is a one to one function, and hence a bijection .
$\frac{x_1-2}{4} = \frac{x_2-2}{4}$
$x_1 = x_2$, is a one to one function. So, $|A| = |B|$.
Solution 2:
(a) $f(x) = \frac{1}{x}$, this is a one to one function from A to B and therefore, $|A| \leq |B|$
(b) $f(x) = 2 + (6 - 2)x$. This is a one to one function from B to A and therefore, $|A| \geq |B|$
Thus, we conclude that $|A| = |B|$.
Note: $f(x) = 2 + (6 - 2)x$ is actually a bijection from $[0, 1]$ to $[2, 6]$ which can be generalized to any $[a, b]$. That is, $[0, 1]$ to $[a, b]$ is given by $f(x) = a + (b - a)x$.

(ii) $A \to B; f(x) = x$. This implies that $|A| \leq |B|$.
$B \to A$; Here, we divide R (i.e., $[-\infty, \infty]$ into four parts and map each to appropriate subinterval in $[0, 1]$ leaving some subinterval in $[0, 1]$).

$f(x) = \frac{1}{2+x}$, if $x \in [1, \infty]$

$f(x) = 0.4 + (0.1)x$, if $x \in [0, 1)$

$f(x) = 0.6 - (0.1)x$ if $x \in (0, -1)$

$f(x) = 0.8 + \frac{-1}{-10+x}$, if $x \in [-1, -\infty]$

Note that the subinterval $(0.5, 0.6)$ does not have a pre-image. This implies that $|B| \leq |A|$. Thus, we conclude that $|A| = |B|$.

(iii) The above mapping can be modified to establish the fact that for any (a, b) or $[a, b]$, their cardinality is same as R. Thus, we can map $[0, 1]$ to R and $(0, 1)$ to R, and therefore $[0, 1] = (0, 1)$.

13. Using PIE, count the number of primes between 2 and 100.
Solution:
Consider the prime factors $2, 3, 5, 7$. P_i represents number of elements in the range 2-100 that are divisible by i.
$|P_i.P_k. \ldots P_j|$ represents the number of elements in the range 2-100 that are divisible by $i \times k \times \ldots \times j$
Number of prime numbers (excluding 2,3,5,7)= 99 - $|P_2|$-$|P_3|$-$|P_5|$-$|P_7|$
+$|P_2P_3|$+$|P_3P_5|$+$|P_5P_7|$+$|P_2P_5|$+$|P_2P_7|$+$|P_3P_7|$-$|P_2P_3P_5|$-$|P_2P_3P_7|$-$|P_3P_5P_7|$
-$|P_2P_5P_7|$+$|P_2P_3P_5P_7|$
$= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0 = 21$
Total number of primes in the range 2-100 = 21+4=25.

14. Using PIE, the number of solutions to $x_1 + x_2 + x_3 = 10$ with $x_1 \leq 2, x_2 \leq 2, x_3 \leq 3$.
Number of solutions = Number of solutions without any constraints - Number of solutions with $(x_1 \geq 3 \vee x_2 \geq 3 \vee x_3 \geq 4)$

Number of solutions without any constraints = Number of reorderings of 10 ones and 2 zeroes =

$12C_2 = 66$
Let A denotes $x_1 \geq 3$, B denotes $x_2 \geq 3$ and C denotes $x_3 \geq 4$.
Number of solutions with $(x_1 \geq 3 \vee x_2 \geq 3 \vee x_3 \geq 4) = n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
$n(A) = n(B) =$ Reordering 7 one's and 2 zeroes (as three one's are already fixed) $= 9C_2 = 36$
$n(C) =$ Reordering 6 one's and 2 zeroes (as four one's are already fixed) $= 8C_2 = 28$
$n(A \cap B) =$ Reordering 4 one's and 2 zeroes (as six one's are already fixed) $= 6C_2 = 15$
$n(A \cap C) = n(B \cap C) =$ Reordering 3 one's and 2 zeroes (as seven one's are already fixed) $= 5C_2 = 10$
$n(A \cap B \cap C) =$ Reordering zero one's and 2 zeroes (as all the 10 one's are fixed) $= 2C_2 = 1$
Number of solutions with $(x_1 \geq 3 \vee x_2 \geq 3 \vee x_3 \geq 4) =$ 36+36+28-15-10-10+1=66
Therefore, Number of solutions = 66-66 =0.

15. The number of matrices of size 3×4 such that the range of each element ($(i,j)^{th}$ entry) is
(i) $\{0, \ldots, k\}$, k is a fixed integer.
There are 12 elements and each element has $(k+1)$ possibilities. Thus, the number of matrices is $(k+1)^{12}$. Finite. (ii) $\{0, \ldots, k\}$, k is a variable integer
. A variable integer is a union of fixed integers, i.e., $k \in \{0, 1, \ldots, \infty\}$. Clearly, k becomes infinity, this counting is same as counting $N \times N \times \ldots \times N$ (12 times), which is countably infinite.

16. The number of matrices of size $k \times l$ such that the range of each element is $[0..m]$, m is a fixed natural number. Solve considering (i) k, l fixed natural number (ii) k, l variable natural number
(i) Finite, $(kl)^{m+1}$ (ii) Countably infinite

17. The number of C programs that are free from syntax errors.
Consider the program P_i that prints i; free from errors. The number of P_i's is countably infinite.

18. The number of C programs that terminate (do not run into infinite loop)
The above example, countably infinite.

19. The number of different algorithms to SORTING.
(i) A_i - the first two elements of the input array are compared i times and for the rest of the elements, follow bubble sort logic. Since i is a natural number, this approach says countably infinite. (ii) A_i: Add i to each element of array, sort using bubble sort, on the sorted array, subtract i from each element of the array. If i is a real number, then there are uncountable logic. Therefore, uncountable.

20. The number of different ways of multiplying a sequence of l matrices, l:fixed number.
Finite. Catalan Number. $\frac{1}{n+1}\binom{2n}{n}$.

21. How many different reflexive binary relations are possible on a set of size n if (i) fixed n (ii) variable n
(i) Finite (ii) Countably infinite

22. The number of functions from $\mathbb{N}$ to $\{1, 2, 3\}$ and the number of functions from $\{1, 2, 3\}$ to $\mathbb{N}$.
(i) Uncountable. Suppose there exists an enumeration. Rows represent functions and column represents the elements of $\mathbb{N}$. The entry $[f_i, a_j] = x$, $x \in \{1, 2, 3\}$. That is, $[f_i, a_j] = 2$, which means, a_j is mapped to '2' in f_i. We now construct a function following diagonalization argument and establish that the function is missed out in the enumeration.
$g(a_i) = 1$, if $[f_i, a_i] = 2$ or 3
$g(a_i) = 2$, otherwise
Clearly, g is a function from the set of natural numbers to the set $\{1, 2, 3\}$ and not listed in the enumeration. Therefore, uncountable.