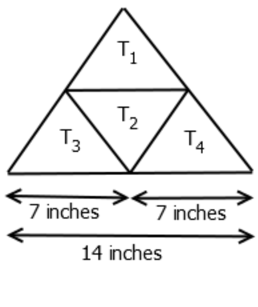
Problem Session 3 - Proof Techniques and Relations

1. King Problem: The king summoned the best mathematicians in the kingdom to the palace to find out how smart they were. The king told them I have placed white hats on some of you and black hats on the others. You may look at, but not talk to one another. I will leave now and will come back every hour on the hour. Every time I return, I want those of who have determined that you are wearing white hats to come up and tell me immediately. As it turned out, at the n^{th} hour every one of the n mathematicians who were given white hats informed the king that she knew that she was wearing a white hat? Why?
`Solution:` We shall prove by mathematical induction on $n \geq 1$, n represents the number of mathematicians with white hats. Base 1: $n = 1$; exactly one mathematician is wearing a white hat and the rest black hats. The person wearing white hat sees all black hats around and hence, at the 1st hour, she reports to the king that she is wearing white. Note that king does not lie and at least one mathematician is wearing white hat for any instance. Base 2: $n = 2$. There are two mathematicians with white hats and we now show that at the end of second hour mathematicians who are given white hats will inform the king about her hat's color. Note that the number of mathematicians is $k \geq 2$, out of which two are wearing white hats and the rest are wearing black hats. Let M_1, M_2 are wearing white and $M_3, \ldots, M_k$ are wearing black. Each $M_i, 3 \leq i \leq k$, sees 2 white hats and $(k-1)$ black hats. Further each M_i thinks that there are at least two 2 white hats as her hat color may be white or black. Both M_1 and M_2 can see one white hat and the rest seen are black hats. For clarity purpose, let us fix M_1. Note that king has placed some white hats (there is no scenario with only black hats). With respect to M_1, had M_1 been wearing black, M_2 would have approached the king at the end of first hour and informed her hat color. The fact that M_2 did not approach the king at the end of first hour will only imply that both M_1 and M_2 are wearing white. Subsequently, they both approach the king at the end of second hour and inform the king that they both are wearing white hats. For clarity purpose, we consider $n = 3$ case also. Let M_1, M_2, M_3 are wearing white and the rest are black. Each black hat person thinks that there are at least 3 white hats. Each of M_1, M_2, M_3 sees two white hats and the rest black hats. Had M_1 been wearing black hat, M_2 and M_3 would have approached the king at the end of second hour, that this does not happen implies that M_1 is wearing white and all three (M_1, M_2, M_3) approach the kind at the end of third hour to inform that they are wearing white hats.

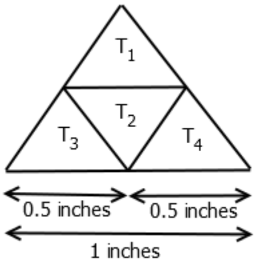
Hypothesis: Assume that there are $n = l$, $l \geq 2$, mathematicians wearing white hats and all report at the end of l^{th} hour that they are wearing white hats.
Induction step: Consider $n = l+1$, $l \geq 2$ mathematicians wearing white hat. Let M_{l+1} be the mathematician wearing white hat and sees l other white hats and the rest are black hats. Had M_1 been black, by the induction hypothesis $M_2, \ldots, M_{l+1}$ would have approached the king at the end of l^{th} hour. Since $M_2, \ldots, M_{l+1}$ did not approach the king at the end l^{th} hour will only imply that M_1 is white and all $M_1, \ldots, M_{l+1}$ approach and inform the king at the end of $(l+1)^{th}$ hour about their hat color.
This complete the proof by induction.

2. Five darts are thrown at an equilateral triangular target measuring 14 inches on a side. Prove that two of them must be at a distance no more than 7 inches apart.
Solution:
Divide the equilateral triangle into four T_1, T_2, T_3 and T_4 equilateral triangles as shown in *Figure.*



Pigeon holes: T_1, T_2, T_3 and T_4
Pigeons: Five darts.
PHP: At least one hole will have two darts and it will be at most 7 inches apart as distance between any two points in any T_i is at most 7.

3. From a bin with 2 red pebbles, 5 green pebbles and 6 blue pebbles, how many must you take to be sure that you have (i) at least 2 colors? (ii) at least 3 colors? (iii) at least 2 of the same color? (iv) at least 4 of the same color?
Solution:



- Seven (6+1). Note: if you pick any 7, you will always find at least 2 colors. For less than 7, some pick may be yes and the rest may be no. Since the pick is arbitrary, 7 is the right answer.

- Twelve (6+5+1).
- Four($1 \in R$+ $1 \in G$+ $1 \in B$ $+1 \in \{R, G, B\}$).
- Nine ($2 \in R$+ $3 \in G$+ $3 \in B$ $+1 \in \{G, B\}$). This implies that in any arbitrary pick of at least 9 pebbles, the claim is true. For less than 9, it may be yes or no.

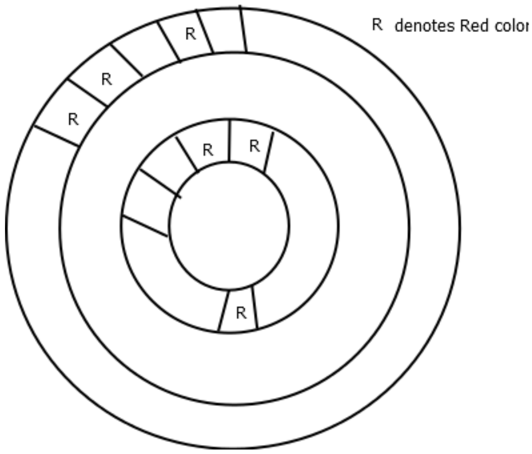
4. Prove that among five points select inside an equilateral triangle with side equal to 1, there always exists a pair at a distance not greater than 0.5.
Solution:
Divide the equilateral triangle into four T_1, T_2, T_3 and T_4 equilateral triangles as shown in *Figure*.
Pigeon holes: T_1, T_2, T_3 and T_4
Pigeons: Five points.
PHP: At least one hole will have two points and it will be at most 0.5 inches apart.

5. Given any set of 7 distinct integers, there must exist 2 integers in this set whose sum or difference is divisible by 7.
Solution:
Pigeon holes: $(0, 7), (1, 6), (2, 5), (3, 4)$, 4 holes.
Pigeons: 7 distinct integers.
PHP: Place the integer x in the hole (y, z) if $x\%7 = y$ or $x\%7 = z$. Note $x\%7$ is x mod 7. There exist at least one hole with at least $\lceil \frac{7}{4} \rceil = 2$ integers such that either both have the same remainder or different remainders. If it has the same remainder then, the difference is divisible by 7. If it has different remainders then, the sum is divisible by 7.

6. Given any set of 7 distinct integers, there must exist 2 integers in this set whose sum or difference is divisible by 10.
Solution:
Pigeon holes: $(0, 0), (1, 9), (2, 8), (3, 7), (4, 6), (5, 5)$, 6 holes.
Pigeons: 7 distinct integers.
PHP: Place the integer x in the hole (y, z) if $x\%10 = y$ or $x\%10 = z$. Note $x\%10$ is x mod 10. There exist at least one hole with at least $\lceil \frac{7}{6} \rceil = 2$ integers such that either both have the same remainder or different remainders. If it has the same remainder then, the difference is divisible by 10. If it has different remainders then, the sum is divisible by 10.

7. Among 61 different integral powers of the integer 5, there are at least 6 of them that have the same reminder when divided by 12.
Solution:
Pigeon holes: $0, 1, \ldots, 11$, there are 12 holes based on possible remainders.
Pigeons: 61 different integral powers of 5.
PHP: Place the number x in the hole y if $x\%12 = y$. There exists at least one hole with at least $\lceil \frac{61}{12} \rceil = 6$ numbers such that they have the same remainder when divided by 12.

8. The circumference of two concentric disks is divided into 200 sections each. For the outer disk, 100 of the sections are painted red and 100 of the sections are painted white. For the inner disk the sections are painted red or white in an arbitrary manner. Show that it is possible to align the two disks so that 100 or more of the sections on the inner disk have their colors matched with the corresponding sections on the outer disk.
Solution:
Fix the outer disk and rotate the inner disk in anti-clock wise direction. Each rotation is referred to as an `orientation` of the inner disk with respect to the outer disk. Let $O_1, \ldots, O_{200}$ denote segments in the outer disk and $I_1, \ldots, I_{200}$ denote segments in the inner disk. Consider the Orientation 1 where O_1 is aligned with I_1, O_2 is aligned with I_2, and



so on. If suppose I_1 is colored white and O_1 is also white, then there is a match. With respect to this orientation, all (I_i, O_i) such that the color of I_i and O_i are same, are designated as match. We now rotate inner disk by one segment so that I_2 is aligned with O_1, I_1 is aligned with O_{200}, I_3 is aligned with O_2, and so on. This is referred to Orientation 2. With respect to this orientation, if color of I_2 and O_1 are same, then they are designated as matching pairs. In general, with respect to a specific orientation, there may be many matching pairs. We shall now count the number of matching pairs. If I_1 is white, then I_1 shall match with 100 white segments of the outer disk in 200 orientations. Similarly, if I_1 is red, then I_1 shall match with 100 red segments of the outer disk in 200 orientations. In general, a white segment on the inner disk shall find a match in 100 orientations out of 200, similarly, a red segment on the inner disk shall find a match in 100 orientations out of 200. That is any segment on the inner disk can match with 100 segments on the outer disk. Overall, for 200 segments on the inner disk, the number of matches is 200 * 100 = 20000 matches.
Consider 200 pigon holes, one for each orientation. We have 20000 pigeons, one for each match. The pigeon hole corresponding to Orientation 1 contain all pairs that match in Orientation 1 and the label Orientation k contain all pairs (I_i, O_j) that match in Orientation k. By pigeon hole principle, we see that if 20000 matches (pigeons) are distributed into 200 orientations (pigeon holes), then there exists a pigeon hole (orientation) in which there are at least $\frac{20000}{200} = 100$ pigeons (matches). The pigeon hole containing at least 100 pigeons gives the orientation in which we get at least 100 matches. Hence, the claim.

9. Suppose that a computer science laboratory has 15 workstations and 10 servers.A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time.We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections. Although we could do this by connecting every workstation directly to every server (using 150 connections), what is the minimum number of direct connections needed to achieve this goal?
Solution:
Let $S_1, S_2, \ldots, S_{10}$ be the 10 servers and $W_1, W_2, \ldots, W_{15}$ be the 15 workstations. Connect $W_1, W_2, \ldots, W_{10}$ to $S_1, S_2, \ldots, S_{10}$, respectively (10 cables). Now, connect each workstation W_{10}, W_{11}, $\ldots$, W_{15} to all 10 servers (50 cables). So that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections and this is minimum to achieve this. Minimum number of direct connections needed to achieve this goal is 60. Suppose, we are given 59 cables. Then, 59 cables (pigeons) are distributed into 10 servers (pigeon holes). By PHP, there exists a server having at most $\lfloor \frac{59}{10} \rfloor = 5$ cables. Say, S_1 has 5 cables incident on it and assume that $W_1 1$ to S_1 cable link is missing and other cables to S_1 are as per the above scheme. Then the request $\{W_2, \ldots, W_1 1\}$ containing 10 service requests can not be serviced despite S_1 being free. A similar argument can be given if any one of the cable is missing in the above scheme. Thus, 60 cables is minimum for this configuration.

10. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers.
Solution:
Let $C_1, C_2, \ldots, C_8$ be the 8 computers and P_1, P_2, P_3, P_4 be the 4 printers. Connect $C_1, C_2, \ldots, C_4$ to P_1, P_2, P_3, P_4, respectively (4 cables). Now, connect computer C_4, C_5, $\ldots$, C_8 to all 4 printers (16 cables). So that for every choice of four of the eight computers, these four computers can directly access four different printers and this is minimum to achieve this. Minimum number of direct connections needed to achieve this goal = 20.

11. Prove that the total number of nodes in a full binary tree (all leaves are at same level) of height h is $2^{h+1} - 1$.
Solution: Induction on h. Base: $h = 0$. The full binary tree of height '0' is just a node and hence, $2^{0+1} - 1 = 1$ is true. Assume a full binary tree of height $h \geq 0$ has $2^{h+1} - 1$. A full binary tree T of height $h + 1$ is obtained from a

full binary tree T' of height h as follows; to each leaf in T' add 2 two leaves so that the height of T' increases by one. Observe that there are 2^h nodes at level h in T'. Thus, T is obtained from T' by adding two leaves to each leaf in T'. Therefore, the total number of nodes in T is $2^{h+1} - 1 + 2 \cdot 2^h = 2^{h+2} - 1$. Thus, the claim follows.

12. What is wrong with this `'proof'`. **Theorem:** For every positive integer n, if x and y are positive integers with $max(x, y) = n$, then $x = y$.

    Basis Step: Suppose that $n = 1$. If $max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$. **Inductive Hypothesis:** Let $k \geq 1$ be a positive integer. Assume that whenever $max(x, y) = k$ and x and y are positive integers, then $x = y$.
    Inductive Step: Now let $max(x, y) = k + 1, k \geq 1$, where x and y are positive integers. Then $max(x - 1, y - 1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.
    Solution: Error is at two places; induction hypothesis and induction step. The inductive hypothesis assumption is false when $k = 2$. For example, $max(1, 2) = 2$, however, $1 \neq 2$. As part of induction step, when $k + 1 = 2$, for instance, $max(1, 2) = 2$. Then, we claim that $max(x - 1, y - 1) = k$, and in our case, it is $max(0, 1) = 1$. Subsequently, we bring in hypothesis and say $x = y$ which is incorrect. The reason is due to the fact that the hypothesis is applicable when both $x-1$ and $y-1$ are integers, and not applicable otherwise. Concluding $max(0, 1) = 1$ is incorrect.

13. `Claim:` All students in DM course get 'S' grade. We now present a proof using mathematical induction on the number of students. `Base:` $n = 1$. 'Renjith' gets 'S' grade. `Hypothesis:` Assume $n = k$ students get 'S' grade. `Induction Step:` Consider a set of $k + 1$ students. The set $\{s_1, \ldots, s_{k+1}\}$ of students contain $\{s_1, \ldots, s_k\}$ and $\{s_2, \ldots, s_{k+1}\}$. Clearly both the sets are of size k and by the hypothesis all students in $\{s_1, \ldots, s_k\}$ get 'S' grade and all students in $\{s_2, \ldots, s_{k+1}\}$ get 'S' grade. Therefore, all students in $\{s_1, \ldots, s_{k+1}\}$ get 'S' grade. This completes the induction. Is the proof correct. If not, identify the flaw in this argument.
    Solution: The flaw occurs at three places ; base, hypothesis and induction step. The flaw in base case; for $n = 1$, the base says 'Renjith' gets 's' grade, however, when $n = 1$, there are many instances for which $n = 1$ is true. For instance, each student in the class satisfies $n = 1$ and base does not say about other $n = 1$ instances. The flaw in hypothesis; assume the claim is true for $n = k$ students, $k \geq 1$, however, the assumption must be with respect to every set of size $k \geq 1$. The flaw in induction step; the set $\{s_1, \ldots, s_{k+1}\}$ is rewritten using two overlapping sets of size k, namely, $\{s_1, \ldots, s_k\}$ and $\{s_2, \ldots, s_{k+1}\}$. However, if $k + 1 = 2$, then there is no overlapping between two sets, which is a gap in the proof.

14. Tray Problem: A tray contains labelled balls and there are finite number of balls on the tray. The game proceeds like this: if you take out a ball labelled $i \geq 2$, you can replace with any number of balls (of course, finite number) whose labels are from $\{1, \ldots, i - 1\}$. There is no replacement for the ball labelled 1. The goal is to show that this game terminates, i.e. there is a sequence of replacements which will result in empty tray.
    Base: $n = 1$. Suppose the tray contains balls labelled '1' only. Clearly, there is a finite sequence of moves which will result in empty tray as there is no replacement for balls that are labelled '1'.
    Hypothesis: $n = k \geq 1$. Let the largest label is k. We assume that there is a sequence of moves which will make the tray empty.
    Induction step: $n = k + 1$, $k \geq 1$. Let $A = \{B_{k+1} \mid$ there is a ball labelled $(k + 1)$ in the tray $\}$. Since there is a replacement for B_{k+1}, start taking each ball labelled B_{k+1} from the tray till A is empty (all B_{k+1} labelled are taken out of the tray). Clearly for each pick, we will replace it with balls labelled $B_1, \ldots, B_k$. Now, in the tray the highest index is k and by the induction hypothesis, there is a sequence which will make the tray empty. This completes the induction and hence the claim.

15. What is the n^{th} Fibonacci number. How many base cases are required for proving this number.
    We assume $F_0 = 0, F_1 = 1$ and $F_2 = F_1+F_0 = 1+0 = 1$. The induction hypothesis assumes that $F_n = F_{n-1}+F_{n-2}, n \geq 2$ and we prove as part of induction step that $F_{n+1} = F_n + F_{n-1}, n \geq 2$. Thus, we need three base cases; F_0, F_1, F_2.

16. Let $R1$ and $R2$ be relations on A. Prove each of the following.

    - $r(R_1 \cup R_2) = r(R_1) \cup r(R_2)$
    - $s(R_1 \cup R_2) = s(R_1) \cup s(R_2)$
    - $t(R_1 \cup R_2) \supset t(R_1) \cup t(R_2)$
    - Show by counter example that $t(R_1 \cup R_2) \not\subset t(R_1) \cup t(R_2)$

    Solution:
    NOTE: $E = \{(x, x) \mid x \in A\}$ and $R^c = \{(a, b) \mid (b, a) \in R\}$.

a. By definition $r(R) = R \cup E$. $r(R_1) = R_1 \cup E$, $r(R_2) = R_2 \cup E$.
$r(R_1) \cup r(R_2) = R_1 \cup E \cup R_2 \cup E = R_1 \cup R_2 \cup E = r(R_1 \cup R_2)$

b. By definition $s(R) = R \cup R^c$. $s(R_1) = R_1 \cup R_1^c$, $s(R_2) = R_2 \cup R_2^c$.
$s(R_1) \cup s(R_2) = R_1 \cup R_1^c \cup R_2 \cup R_2^c = R_1 \cup R_2 \cup (R_1 \cup R_2)^c = s(R_1 \cup R_2)$

c. $R_1 \subset t(R_1 \cup R_2)$. For every $(a,b),(b,c) \in R_1$, $(a,c) \in t(R_1)$. It follows that $(a,c) \in t(R_1 \cup R_2)$. Similar arguments hold for R_2. Therefore $t(R_1) \cup t(R_2) \subset t(r_1 \cup R_2)$

d. $A = \{1,2,3\}, R_1 = \{(1,2)\}, R_2 = \{(2,3)\}$
$t(R_1 \cup R_2) = \{(1,2),(2,3),(1,3)\}$, $t(R_1) = \{(1,2)\}$, $t(R_2) = \{(2,3)\}$ and $t(R_1) \cup t(R_2) = \{(1,2),(2,3)\}$
here $t(R_1 \cup r_2) \not\subset t(R_1) \cup t(R_2)$

17. Construct examples of the following sets:

- A non-empty linearly ordered set (total order) in which some subsets do not have a least element.
- A non-empty partially ordered set which is not linearly ordered and in which some subsets do not have a greatest element. Construct both finite and infinite examples.
- A partially ordered set with a subset for which there exists a glb but which does not have a least element. Construct both finite and infinite examples.
- A partially ordered set with a subset for which there exists an upper bound but not a least upper bound. Construct both finite and infinite examples.

Solution:
(a) $(I, \leq)$
(b) **Example: Finite set** The poset given in Figure 1 is not a linearly ordered set and the subset $\{d, e\}$ does not

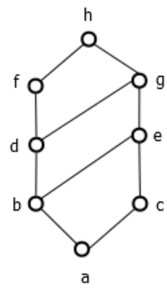


Figure 1:

have the greatest element.

Example: Infinite Set $(N, |)$, where $a \mid b$ denotes a divides b. The set itself does not have the greatest element.

(c) **Example: Finite set** The poset given in Figure 1, has a subset $\{d, e\}$ for which there exists a glb, $\{b\}$, but which does not have a least element.

Example: Infinite Set $(N, |)$, where $a \mid b$ denotes a divides b. The subset $\{4, 6\}$ has a glb, $\{2\}$, but does not have a least element.
(d) **Example: Finite set** The poset given in Figure 2, has a subset $\{a\}$ for which there exists a upper bound,

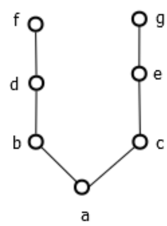
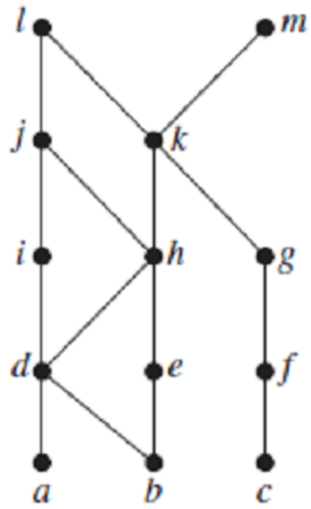


Figure 2:

$\{b, c, d, e, f, g\}$ but no least upper bound.



Example: Infinite set Set: R, Subset: $(0, 1)$, Relation: less than. Upper bound $\{x \mid x \geq 1\}$, however the subset has no least upper bound.

18. For the following hasse diagram, find (a) Find the maximal elements. b) Find the minimal elements. c) Is there a greatest element? d) Is there a least element? e) Find all upper bounds of $\{a, b, c\}$. f) Find the least upper bound of $\{a, b, c\}$, if it exists. g) Find all lower bounds of $\{f, g, h\}$. h) Find the greatest lower bound of $\{f, g, h\}$, if it exists. (a) Find the maximal elements.
Solution: $\{l, m\}$

b) Find the minimal elements.
Solution: $\{a, b, c\}$

c) Is there a greatest element?
Solution: No

d) Is there a least element?
Solution: No

e) Find all upper bounds of $\{a, b, c\}$.
Solution: $\{k, l, m\}$

f) Find the least upper bound of $\{a, b, c\}$, if it exists.
Solution: $\{k\}$

g) Find all lower bounds of $\{f, g, h\}$.
Solution: NIL

h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.
Solution: NIL

19. Let R be a transitive and reflexive relation on A. Let T be a relation on A, such that (a, b) is in T iff both (a, b) and (b, a) are in R. Show that T is an equivalence relation.
Solution: Since R is reflexive, for each $a \in A, (a, a) \in R$. Since $(a, b = a)$ and $(b = a, a)$ are in R, $(a, a) \in T$ for each $a \in A$. Therefore, T is reflexive.
If (a, b) in T, then both (a, b) and (b, a) are in R. This implies, $(b, a) \in T$. Thus, T is symmetric.
If $(a, b), (b, c) \in T$, then both (a, b) and (b, a) are in R and both (b, c) and (c, b) are in R. Since R is transitive, $(a, c), (c, a) \in R$. Since $(a, c), (c, a) \in R$, it follows that $(a, c) \in T$. Therefore, T is transitive. Thus, we conclude T is an equivalence relation.

20. Find a set A with n-elements and a relation R on A such that $R^1, R^2, \ldots, R^n$ are all distinct. This establishes the bound $t(R) = \cup_{i=1}^{n-1} R_i$.
Solution: $A = \{1, 2, \ldots, n\}$ and $R = \{(i, i+1) \mid 1 \le i \le n-1\}$. $R^1 = R$. $R^i = \{(a, c) \mid (a, b) \in R^{i-1} \wedge (b, c) \in R^1\}$. If we focus on the underlying graph theoretic representation of R, then R^2 introduces edges between nodes which are at a distance two in R^1. That is, for each i, the edges between $(i, i+2)$. Finally, R^n introduces edges between $(i, i+n-1)$. Further, R^1 is not transitive, $R^1 \cup R^2$ is not transitive, $R^1 \cup \ldots \cup R^{n-2}$ is not transitive and $\cup_{i=1}^{n-1} R_i$ is transitive. Therefore, $t(R) = \cup_{i=1}^{n-1} R_i$.

21. Are there non well-ordered sets on finite sets. That is, is there a finite set A, $R \subseteq A \times A$ such that R is a total order and there exists $(\neq)A' \subseteq A$, A' has no least element.
Solution: Since any well-order is a total order and any total order on finite sets is a path on n nodes (the underlying Hasse diagram). Observe that any non-empty subset of a total order is just a subpath in the path graph and any subpath has a least element. Therefore, every finite total ordered set is also a well ordered set.

22. Identify a set A and a total order relation R on A such that for some subset A' there is no least element. That is, A is not a well-ordered set.
Solution: From the above solution, it is clear that A can not be a finite set. Consider $(\mathbb{I}, \le)$. This set is a total order but not a well order as there is no least element.

23. How many irreflexive transitive relations are there on a set of size n.

Observe that irreflexive transitive relations are antisymmetric. Thus, we are interested in counting relations which are irreflexive, transitive and antisymmetric. Partial order relations satisfy reflexive, transitive and antisymmetry properties. This implies that, partial order relations with out reflexive pairs are essentially relations that satisfy irreflexive, transitive and antisymmetry properties. Therefore, counting partial orders and counting relations that satisfy irreflexive, transitive and antisymmetric properties are same. Refer to Q25 for counting partial orders.

24. Let (A, R) be a poset and B a subset of A.
   - If b is a greatest element of B, then b is a maximal element of B.
   - If b is a greatest element of B, then b is lub of B.

Solution:
a. An element $b \in B$ is a *greatest element* of B if for every $b' \in B, b' \preceq b$. An element $b \in B$ is a maximal element of B if $b \in B$ and there does not exist $b' \in B$ such that $b \neq b'$ and $b \preceq b'$. Therefore if b is a greatest element, then there does not exist $b' \in B$ such that $b \neq b'$ and $b \preceq b'$, implies that b is a maximal element.

b. An element $b \in A$ is an upper bound for B if for every element $b' \in B$, $b' \preceq b$. An element $b \in A$ is a least upper bound (lub) for B if b is an upper bound and for every upper bound b' of B, $b \preceq b'$. Therefore, if b is a greatest element, then b is clearly an upper bound. Since $b \in B$, it must be the case that $b \preceq b'$ for every upper bound b'. Therefore, b is *lub*.

25. How many different posets, totally ordered, well-ordered relations are there on a set of size n.
Solution:
total order: Since total order on n nodes is a path graph and there are $n!$ paths on n nodes, the number of total order relations is $n!$.

well order: Since any total order on a finite set is a well order, there are $n!$ well orders.

partial order: note that partial order iff Hasse diagram. Further, Hasse diagram iff the underlying graph is undirected triangle free. Therefore, counting partial orders is equivalent to counting triangle free undirected graphs. The total number of graphs minus graphs with triangles yields the desired number. Graphs with triangles; fix three edges and together with any subset from $\binom{n}{2} - 3$ gives triangle graphs. Therefore, the count is $2^{\binom{n}{2}} - 2^{\binom{n}{2}-3}$.

26. Using PIE (principle of inclusion and exclusion), Find the number of positive integers not exceeding 100 that are either odd or the square of an integer
Solution:
Number of odd numbers = $|O| = 50$

Number of square numbers = $|S| = 10$
Number of odd square numbers = $|O \cap S| = 5$
$|O \cup S| = |O| + |S| - |O \cap S|$
$= 50 + 10 - 5 = 55$

27. Construct a bijection from A to B

a. $A = I, B = N$

b. $A = N, B = N \times N$

c. $A = [0, 1), B = (\frac{1}{4}, \frac{1}{2}]$

d. $A = R, B = (0, \infty)$

Solution:
a. $f(x) = 2|x|$ if $x \geq 0$
$f(x) = 2|x| + 1$ if $x < 0$
b. Diagonalisation argument.
c. $f(x) = \frac{2-x}{4}$
d. $f(x) = e^x$

28. Let (A, R) be a poset and B a subset of A. Prove the following

a. If b is a greatest element of B, then b is a maximal element of B

b. If b is a greatest element of B, then b is lub of B

Solution:
a. An element $b \in B$ is a *greatest element* of B if for every $b' \in B, b' \preceq b$. An element $b \in B$ is a maximal element of B if $b \in B$ and there does not exist $b' \in B$ such that $b \neq b'$ and $b \preceq b'$. Therefore if b is a greatest element, then b is a maximal element.
b. An element $b \in A$ is upper bound for B if for every element $b' \in B$, $b' \preceq b$. An element $b \in A$ is a least upper bound (lub) for B if b is an upper bound and for every upper bound b' of B, $b \preceq b'$. Therefore, if b is a greatest element, then b is also a lub.