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COM205T Discrete Structures for Computing-Lecture Notes

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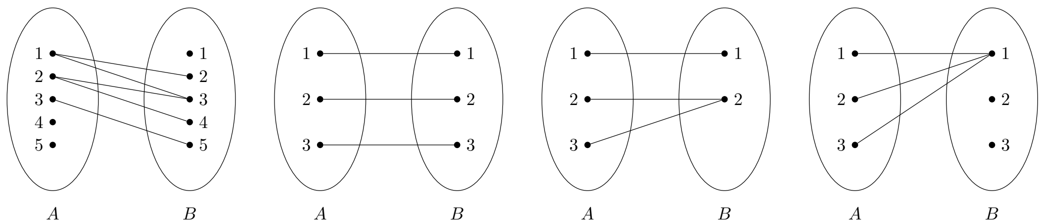
Functions and Infinite Sets

Objective: We shall introduce functions, special functions and related counting problems. We shall see the importance of functions in the context of infinite sets and discuss infinite sets in detail. Further, we introduce the notion counting in the context of infinite sets.

Revisit: Relations

Consider the following relations and its graphical representation, and this representation is different from the one we discussed in the context of relations. $A = B = \{1, 2, 3, 4, 5\}$, $R_1 = \{(1, 2), (1, 3), (2, 3), (2, 4), (2, 5)\}$
$A = B = \{1, 2, 3\}$, $R_2 = \{(1, 1), (2, 2), (3, 3)\}$
$A = B = \{1, 2, 3\}$, $R_3 = \{(1, 1), (2, 2), (2, 3)\}$
$A = B = \{1, 2, 3\}$, $R_4 = \{(1, 1), (2, 1), (3, 1)\}$
We define special relations; relations satisfying the following properties.



(a) R_1 (b) R_2 (c) R_3 (d) R_4

Fig. 1: (a),(b),(c),(d) represent relations using two partitions.

1. **Property P1:** $\forall a \in A\ \exists! b \in B\ [(a, b) \in R]$
2. **Property P2:** relation satisfying P1 and $\nexists a \in A\ \nexists b \in A\ \wedge\ a \neq b\ \wedge\ \forall c \in B\ [(a, c) \in R \wedge (b, c) \in R]$
3. **Property P3:** relation satisfying P1 and $\forall b \in B\ \exists a \in A[(a, b) \in R]$

   A binary relation R satisfying $P1$ is known as a *function* in the theory of relations. Similarly, the one satisfying P2 is known as one-one functions and the one satisfying P3 is known as onto functions.

   Definition 1 (Function) *Let A and B be non-empty sets. A* function *or a* mapping *f from A into B, written as $f : A \to B$, where every element $a \in A$ is mapped to a unique element $b \in B$.*

Definition 2 (One-one function/Injective) *A function is said to be injective if for every element in the range there exists a unique pre-image. i.e., no two elements in the domain map to same element in the co-domain.*

Definition 3 (Onto function/Surjective) *A function is said to be surjective if for every element in co-domain there exists a pre-image.*

Definition 4 (Bijective) *A function is said to be* bijective *if it is both one-one and onto function.*

From the Figure 1, we can observe that

(a) is not satisfying property P1 because elements 1 and 2 in A are mapped to two elements in B which is a forbidden structure according to property P1 and elements 4 and 5 in A is not mapped to any element in B. Since R_1 is not satisfying the property P1, it does not satisfy P2 and P3 as well. Therefore, (a) is not a function.

(b) is satisfying properties P1, P2 and P3.

(c) is satisfying properties P1 and P3 but not P2 because 2 and 3 in A are mapped to same element in B which is forbidden according P2.

(d) satisfies Property P1 but not P2 because $1, 2, 3$ in A is mapped to same element in B which is forbidden according to P2 and it does not satisfy P3 because there exist elements 2 and 3 in B which is not mapped to any elements in A which is forbidden according to property P3.

Note:
1. The element $b \in B$ is called the *image* of a under f and is written as $f(a)$.
2. If $f(a) = b$ then a is called the *pre-image* of b under f.
3. A is called the *domain* of f and $f(A) = \{f(a) \mid a \in A\}$ is called the *range* of f. B is called the *co-domain* of f.
4. Examples of functions:
   a) Domain = Set of apples and Range = weight of apples.
   b) Domain = Set of students and Range = CGPA of students.
5. Every function is a relation but the converse is not true.
6. For every element $a \in A$, there exists an image $f(a)$. i.e., $f(a)$ is well defined.
7. For one-one functions, $|B| \geq |A|$ and for onto functions $|A| \geq |B|$.
8. For functions that satisfy both one-one and onto, $|B| = |A|$.

Things to know:
1. Is there a function for which $A = \emptyset$ and $B \neq \emptyset$? Yes, an empty function.
2. Is there a function for which $A \neq \emptyset$ and $B = \emptyset$? No.
3. Is there a function for which $A = \emptyset$ and $B = \emptyset$? Yes, void function.

- Let $A = B = N$(set of natural numbers).
  - Give an example of a function $f : N \to N$, which is one-one and not onto ? $f(x) = x+1$
  - Give an example of a function $f : N \to N$, which is not one-one but onto ? $f(x) = \lceil \frac{x}{2} \rceil$
  - Give an example of a function $f : N \to N$, which is one-one and onto ? $f(x) = x$

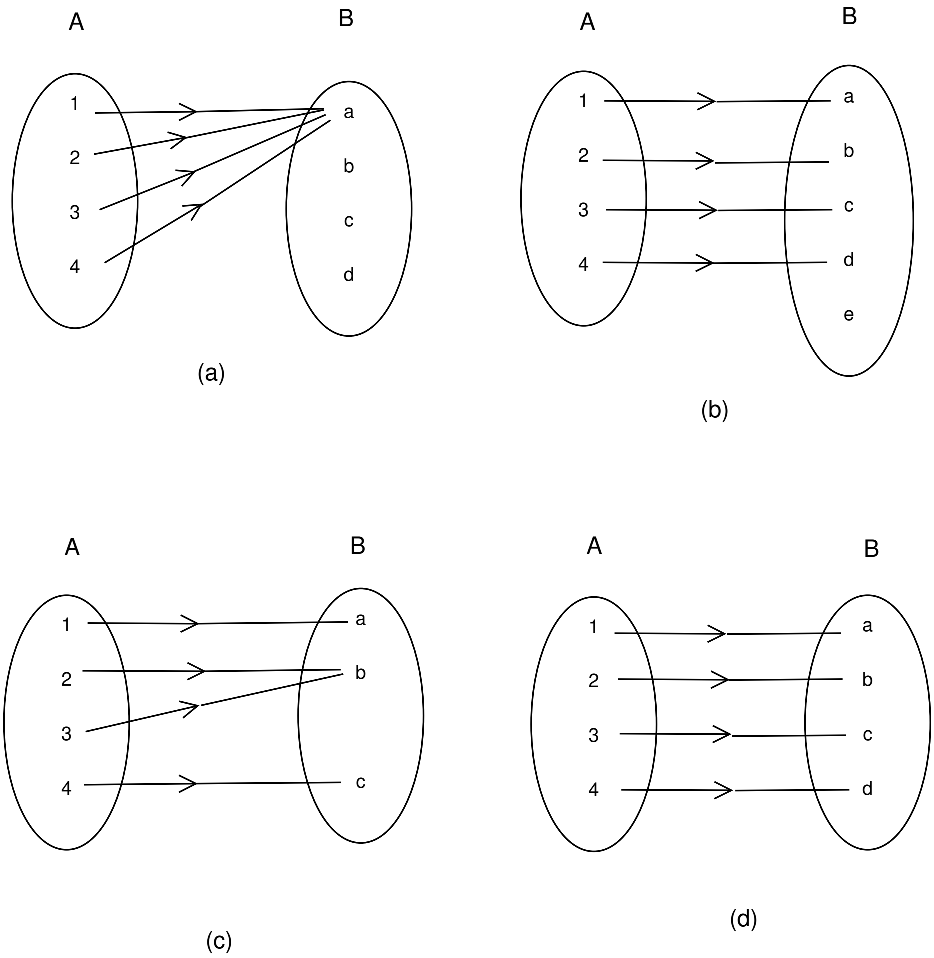
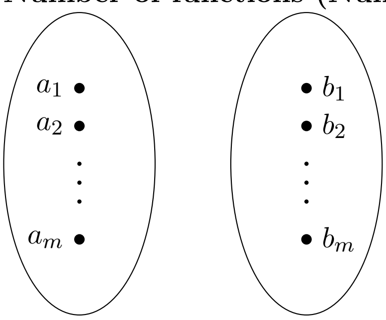


Fig. 2: (a) Not one-one and not onto (b) one-one but not onto (c) onto but not one-one (d) one-one and onto

- Let $A = B = I$(set of integers).
  - Give an example of a function $f : I \to I$, which is one-one and not onto ? $f(x) = 2x$
  - Give an example of a function $f : I \to I$, which is not one-one but onto ? $f(x) = \lceil \frac{x}{2} \rceil$.
  - Give an example of a function $f : I \to I$, which is one-one and onto ? $f(x) = x$
- Give an example bijective function $f : (0, 1) \to (4, 5)$ in real numbers ? $f(x) = x + 4$.

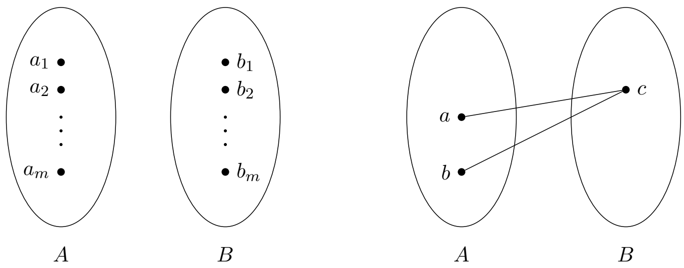
Counting functions:

Consider two sets A and B and a relation $R \subseteq A \times B$ and $|A| = n,\ |B| = m$.
Number of functions (Number of R satisfying P1):



The element a_1 in A can be mapped to m elements in B, a_2 in A can also be mapped to m elements in B. Similarly each element in A can be mapped to m elements in B. Since each element is having m choices to be mapped. Therefore,
Number of functions from A to B =Number of Relations $R \subseteq A \times B$ satisfying P1
$= m \times m \times m \times \ldots \times m (n \text{ times}) = m^n$ or in general, $|B|^{|A|}$.

Number of one-one functions (Number of R satisfying P2):



(a) R (b) forbidden structure for P_2

For property P2, note that $m \geq n$, otherwise one-one function does not exist.
The element a_1 has m possibilities, the element a_2 has $(m-1)$ possibilities, $(m-1)$ because except the element in B which was mapped to a_1, similarly a_n will have $(m-(n-1))$ possibilities.

a_1	a_2	a_3	$\ldots$	a_n
has m possibilities	has $m-1$ possibilities	has $m-2$ possibilities	$\ldots$	has $m-(n-1)$
$b_1, b_2, \ldots, b_m$	suppose b_i has been $B \setminus \{b_i\}$	suppose b_j has been selected	$\ldots$	$\ldots$
	selected for a_1 then	for a_2 then $B \setminus \{b_i, b_j\}$		

$= m \times m-1 \times m-3 \times \ldots \times m-(n-1)$

$= \dfrac{m \times m-1 \times m-3 \times \ldots \times m-(n-1) \times (m-n) \times \ldots \times 3 \times 2 \times 1}{(m-n) \times \ldots \times 3 \times 2 \times 1}$

$= \dfrac{m!}{(m-n)!}$

$= m_{P_n} = n! \times m_{C_n}$

• Let $|A| = n$ and $|B| = m$. The number of different one-one functions are $m_{C_n} \times n! = m_{P_n}$.

Remarks:
1. Note that any one-one function from A to B is a bijective function from A to $f(A)$.
2. Counting one-one functions is equivalent to selecting n elements out of m elements from codomain.
3. Counting one-one functions is equivalent to choosing n elements out of m elements from codomain and considering all possible bijective functions on a set of size n.
4. Let $|A| = n$ and $|B| = m$. If $X = \{R \mid R$ is a relation defined w.r.t $(A, B)\}$ and $Y = \{R \mid R$ is a function w.r.t $(A, B)\}$ then, $|X| \geq |Y|$ (Since, $|X| = 2^{mn}$ and $|Y| = m^n$).

Number of onto functions (Number of R satisfying P3):

We shall next count the number of onto functions using PIE.

Let $\mid A \mid= n$ and $\mid B \mid= m$.
Number of onto functions = Number of functions - Number of functions that are not onto.

The number of functions from A to B = m^n
Let $B = \{a_1, a_2, \ldots\}$. The number of functions between A and B in which there is no pre-image for $a_1 = (m-1)^n$. Similarly, for $a_2, a_3, \ldots$

Therefore, the total number of functions in which there is no pre-image for $a_i = m_{c_1}(m-1)^n$.

Similarly, the total number of functions in which there is no pre-image for $(a_1, a_2) = (m-2)^n$. Therefore, for any two $(a_i, a_j) = m_{c_2}(m-2)^n$.

The function not having pre-image for both a_1 and a_2 is counted thrice in the above counting. Once with respect a_1 (function not having pre-image for a_1), counted again with respect to a_2 (function not having pre-image for a_2) and counted for the third time when seen as a function not having pre-image for both (a_1, a_2). However, the function has to be counted exactly once. Due to this reason, we employ principle of inclusion and exclusion and hence, the count is $2 - 1 = 1$.

Thus, the number of onto functions =
$$m^n - \left[m_{c_1}(m-1)^n - m_{c_2}(m-2)^n + m_{c_3}(m-3)^n - \ldots + (-1)^{m-1} m_{c_{m-1}}(1)^n\right]$$

An invitation to infinite sets

Motivation:

- Many interesting sets such as
  - (i) Set of prime numbers
  - (ii) Set of C-programs
  - (iii) Set of C-programs with exactly three statements

  are infinite in nature.
- Between I and N, which set is bigger ?
- Between I and $N \times N$, which set is bigger ?
- Is $[0, 1]$ is bigger than R ?
- Can we list all C-programs ?

Definition 5 (Finite) *A set A is finite if there exists a $n \in N$, n-represents the cardinality of A such that there is a bijection $f : \{0, 1, \ldots, n-1\} \to A$. A is infinite if A is not finite.*

Definition 6 (Infinite) *Let A be a set. If there exists a function $f : A \to A$ such that f is an injection and $f(A) \subset A$ then A is infinite.*

• Let B be a finite set, $B = \{1, 2, \ldots, 10\}$. Can you establish a 1-1 function $f : B \to B$ such that $f(B) \subset B$? No.

• Can you establish a 1-1 function $f : N \to N$ such that $f(N) \subset N$? Yes, $f(x) = x + 1$.

• Every 1-1 function from $f : B \to B$ is also a bijection from $B \to B$ if B is finite.

Problem 1: Show that N is infinite.

Proof by contradiction: Suppose N is finite. By definition, there exists n, n represents the cardinality of N such that $f(n) = a_n$. Let $K = MAX\{f(0), f(1), \ldots, f(n-1)\} + 1$. There does not exist a $x \in \{0, 1, \ldots, n-1\}$ such that $f(x) = k$.
Therefore, f is not onto and thus, f is not a bijection. Hence, our assumption is wrong and N is infinite.

Problem 2: Show that R is infinite.

Proof: To prove R is infinite, establish a function $f : R \to R$ such that f is 1-1 and $f(R) \subset R$.
Consider $f : R \to R$ such that $f(x) = x + 1$, if $x \geq 0$ and $f(x) = x$, if $x < 0$. This function f is 1-1 but not onto (Since, 0 does not have a pre-image) and $f(R) \subset R$. Hence, R is infinite.

Problem 3: Let $\sum = \{a, b\}$ and $f : \sum^* \to \sum^*$. Show that $\sum^*$ is infinite.

Proof: To prove $\sum^*$ is infinite, establish a function $f : \sum^* \to \sum^*$ such that f is 1-1 and $f(\sum^*) \subset \sum^*$.
Consider $f : \sum^* \to \sum^*$ such that $f(x) = ax$. The elements $\epsilon, b, ba, bb, \ldots$ will not have a pre-image. Therefore f is not onto but 1-1 and $f(\sum^*) \subset \sum^*$. Hence, $\sum^*$ is infinite.

Problem 4: Show that $[0, 1]$ is infinite.

Proof: Consider $f : [0,1] \to [0,1]$ such that $f(x) = \frac{x}{2}$. This function f is 1-1 but not onto (Since, $(\frac{1}{2}, 1]$ does not have a pre-image) and $f([0,1]) \subset [0,1]$. Hence, $[0,1]$ is infinite.

Claim: 1 Let A' be a subset of A. If A' is infinite then A is infinite.

Proof: Given A' is infinite. Therefore, there exists a function $g : \; A' \to A'$ such that g is 1-1 and $g(A') \subset A'$.
Consider a function $f : A \to A$ such that

$f(x) = x$, if $x \in A \backslash A'$
$f(x) = g(x)$, if $x \in A'$.

The function f is also 1-1 and $f(A) \subset A$ (Since, g is 1-1 and $g(A') \subset A'$). Thus, A is infinite.

Corollary: Every subset of a finite set is a finite set.

Claim 2: Let $f : A \to B$ be an injection. If A is infinite then B is infinite.

Proof: Since f is 1-1 and A is infinite, $f(A)$ is infinite. By previous claim, B is infinite (since, $f(A) \subseteq B$).

Another proof: In this proof, we shall present the mapping as well. Since $f : A \to B$ is one-one, f is bijective between A and $f(A)$. Note that inverse of a function f exists iff f is bijective. Thus, there exists an inverse f^{-1} from $f(A)$ to A. Further, there exists one-one function $h : A \to A$ such that $h(A) \subset A$. We shall now present a one-one function $g : B \to B$ such that $g(B) \subset B$.

$g(x) = x$ if $x \in B \setminus f(A)$.
$g(x) = f \circ h \circ f^{-1}(x)$ if $x \in f(A)$.

Note that $f \circ h \circ f^{-1}$ is one-one from B to B. Further, $f \circ h \circ f^{-1}(B)$ is $f \circ h(A)$. Since h is one-one, $h(A) \subset A$ and f is also one-one, therefore, $f \circ h \circ f^{-1}(B) \neq B$. Therefore, g is a function from B to B such that $g(B) \subset B$.

Problem 5: A is infinite. Show that (i) $P(A)$, power set of A is infinite. (ii) $A \cup B$ is infinite. (iii) $A \times B$ is infinite. (iv) A^B, the set of all functions from B to A, is infinite.

Solutions:
(i) Consider a function $f : A \to P(A)$ such that $f(x) = \{x\}$. The function f is 1-1 but not onto i.e.,$A \subseteq P(A)$. Thus, $P(A)$ is infinite. (by claim 2)

(ii) We know that, $A \subset A \cup B$. Since A is infinite, $A \cup B$ is infinite (by claim 1).

(iii) Consider a function $f : A \to A \times B$ such that $f(x) = (x, a)$ for some $a \in B$. Clearly, the function f is 1-1 but not onto. Thus, $A \times B$ is infinite. (by claim 2)

(iv) Every element in A^B is a function from $B \to A$. Consider a function $f : A \to A^B$

such that $f(x) = g$, g is a function from $B \to A$ such that $g(b) = x$, $\forall\ b \in B$. This function f is 1-1 but not onto. Thus, A^B is infinite. (by claim 2)

Definition 7 (Countable) *A set A is countable if A is finite or if A has an enumeration (Listing elements of A) or if there exists a bijection from N to A.*

• A set A is said to be Countably infinite if there exists a bijection from N to A or if there exists an enumeration.
• Countable sets are either countably finite or countably infinite.
• $\mid A \mid = \mid B \mid$ if and only if there exists a bijection from A to B.
• $|A| = |B|$ iff there exists an injection from A to B and there exists an injection from B to A.

Problem 6: Prove: $\mid N \mid = \mid I \mid$
Solution 1: Consider the function $f : N \to I$ such that $f(x) = \frac{-(x+1)}{2}$, if $x = 2k+1$, for some integer k and $f(x) = \frac{x}{2}$, if $x = 2k$, for some integer k. Clearly, this function is 1-1 and onto. Thus, $\mid N \mid = \mid I \mid$. We assume that the natural number set has '0' in it and it is mapped to '0' in the integer set.

Problem 7: Prove that the set P of prime numbers is infinite.
Solution: We know that for every prime number x, there exists a prime number $y > x$ (refer to PMI scribe). Therefore, we establish a map between N and the set of prime numbers. $f : N \to P$ such that $f(i) = P_i$. i.e., i^{th} natural number maps to i^{th} prime number P_i. Therefore, there exists an enumeration and hence P is countably infinite.

Problem 8: Prove that the set of positive rational numbers, Q^+, is infinite.
Solution: We below enumerate(list) elements of Q^+ in a systematic way as illustrated in the Figure. We then establish a bijective function from N to Q^+ by following the arrows as illustrated in the figure. This yields an enumeration and a mapping to N, therefore Q^+ is infinite. Thus, Q^+ is countably infinite.

Also, it is clear that Q^- has a bijective mapping to N.

Problem 9: Prove that the cardinality of $\sum^*$ is countably infinite.
Solution: We shall establish this claim by listing the elements of $\sum^*$ using standard ordering. i.e., we list length one strings, followed by length two strings, and so on. Strings having same length will be listed as per lexicographic ordering. An illustration as per standard ordering is shown below.

Problem 10: Prove that the number of C-programs is countably infinite.
Solution: Let $\sum = \{a, b, \ldots, z, A, B, \ldots, Z, \$, \{, \}, \backslash, \ldots\}$ and $\sum^* = \{\epsilon, String1, String2, \ldots\}$. Note that $\sum$ is precisely the set of keys available in a key board (ASCII characters). It is easy to see that every C-program is an element in $\sum^*$, i.e, imagine the case where we write a C-program in horizontal fashion instead of vertical fashion. Also, we do not concerned about whether the C-program is syntactically correct or not, will the program halt, whether it is a meaningful program, etc. We know that $\sum^*$ is countably infinite, thus, the number of C-programs is countably infinite. That is, one can associate a C program with every element

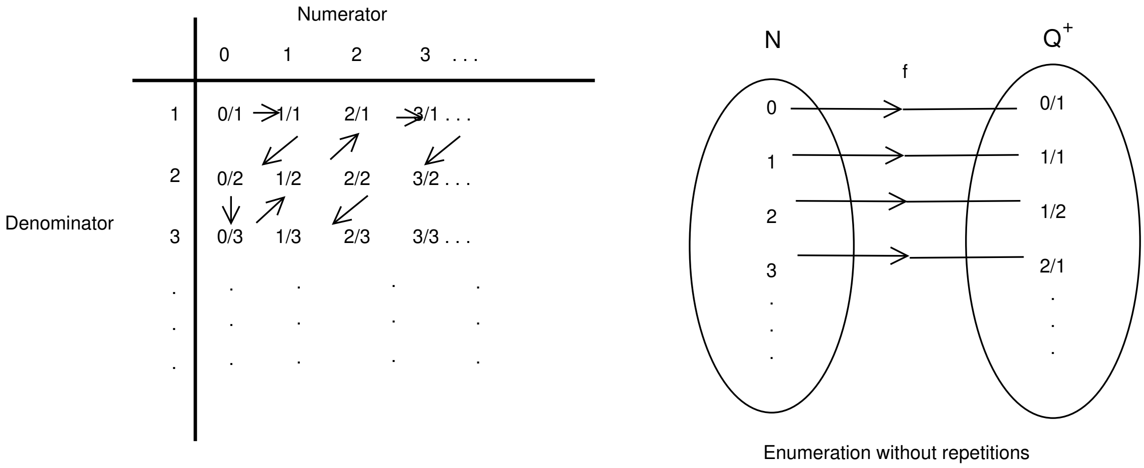


Fig. 3: Enumeration of Q^+

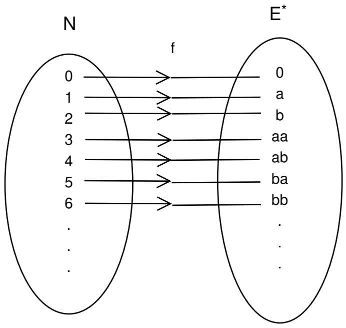


Fig. 4: Enumeration and a bijective function from N to $\sum^*$

of $\sum^*$.

Problem 11: Prove that the number of C-programs with exactly 3 statements is countably infinite ?
Solution:
Let program-1 be read x; $x = x + 1$; print x;, program-2 be read x; $x = x + 2$; print x;, $\ldots$, program-∞ be read x; $x = x + \infty$; print x;. Clearly, the set of programs (Program-1,...) is infinite. We further know from the previous claim that the number of C-programs is countably infinite. Since this set is only a subset of all C-programs, implies that, the number of C-programs with exactly 3-statements is countably infinite.

Problem 12: Prove that $N \times N$ is countably infinite ?
Solution: Consider an infinite matrix with rows and columns representing the elements of N. The row i and column j represents the element $(i, j) \in N \times N$. The argument similar to Q^+ can be given to enumerate the elements of $N \times N$ and associate them to the corresponding natural number. Hence, $N \times N$ is countably infinite.

Remarks:
1. One can establish $N \times N \times N$ is countably infinite.
2. $I \times N$ is countably infinite and any rational number of the form $\frac{p}{q}$ is an element $(p, q) \in I \times N$. Therefore, the set of rational numbers is countably infinite.
3. For all countable sets, there exists an ordering and hence they are well ordered sets.
4. The principle of mathematical induction can be applied to any countable set as the notion successor is well defined.

5. $(Q^+, \leq)$, $(Q^-, \geq)$ are well ordered sets.

Uncountable Sets

In earlier sections, we introduced infinite sets and techniques for showing a set is infinite. Further, we also presented an approach to count infinite sets (countably infinite). We now ask; is every set countable, i.e., either countably finite or countably infinite. We answer this in negative and show that there are uncountable sets. We next introduce cantor's famous technique, diagonalization technique using which we show that the set $[0, 1]$, the set of real numbers between 0 and 1 is uncountable.

Problem 12: Show that $[0, 1]$ uncountable
Solution: We present a proof by contradiction. Suppose $[0, 1]$ is countable then there exists an enumeration, i.e., listing of elements in $[0, 1]$ in a systematic way; ENUM: $x_1, x_2, \ldots$. Further, there exists a bijection from N to ENUM. Let

$$\begin{aligned} x_1 &= 0.x_{11}x_{12}x_{13}x_{14}\ldots \\ x_2 &= 0.x_{21}x_{22}x_{23}x_{24}\ldots \\ x_0 &= 0.x_{31}x_{32}x_{33}x_{34}\ldots \\ &\ldots \end{aligned}$$

We now show that the above listing is incomplete by exhibiting an element $y \in [0, 1]$ and y is not listed in ENUM. Consider $y = 0.y_1y_2y_3y_4\ldots$, such that $y_i = 3$, if $x_{ii} = 2$ and $y_i = 2$, if $x_{ii} \neq 2$. Clearly, $y \in [0, 1]$ and any x_i and y will differ at one position (at least one, position i itself). Therefore, y is not enumerated in $x_1, x_2, \ldots$. Thus, ENUM is incomplete and our assumption that $[0, 1]$ is countable is wrong. Therefore, $[0, 1]$ uncountable.
Note:
1. Since $[0, 1]$ is uncountable, the set of real numbers R is uncountable. The above argument can be extended for any real line $[a, b]$ to show that $[a, b]$ is uncountably infinite.
2. For uncountable sets, there does not exist an enumeration, i.e., there is no notion of first element, second element, etc.

Claim: Countable union of countable sets is countable.

Proof: Let $A_1, A_2, \ldots,$ be a set of countable sets. Since each set A_i is countable, then there exists an enumeration of A_i. Construct a matrix with the first row listing the elements of A_1, the second row listing the elements of A_2, and so on. Now, similar to the proof showing the set of rational numbers is countable, enumerate the elements of the constructed matrix to get a natural mapping to the set of natural numbers. Therefore, the claim follows.

Problem 13: Is irrational numbers countable ?.
Solution:
Assume that irrational numbers are countable. Since rational numbers are countable and by the above claim, union of rational numbers and irrational numbers is countable, however, this is precisely the set of real numbers, which is a contradiction. Thus, the set of all irrational numbers is uncountable.

Remarks:
1. For all countably infinite sets $(N, Q^+, \sum^*, ...)$, the set consists of finite strings. That is, an infinite set of finite strings.
2. For irrational numbers, there is no decimal expansion that terminates. This implies that the expansion is always infinite. The set of irrational numbers is an example infinite set in which each string is infinite.
3. For all uncountable sets, there exists a subset which is infinite containing infinite length strings.
4. The set of C programs is an example infinite set in which each program is of finite length and hence, an element of $\sum^*$.

Problem 14: Given that $\sum^*$ is countably infinite. Is $P(\sum^*)$ countably infinite ?
Solution: Suppose $P(\sum^*)$ is countably infinite.
Since $\sum^*$ is countably infinite we can enumerate $\sum^*$ as $x_1, x_2, x_3, \ldots$. Since $P(\sum^*)$ is countably infinite, there exists an enumeration $A_1, A_2, A_3, \ldots$, where each A_i is subset of $\sum^*$. We now construct a matrix with row representing the sets (A_i's) and column representing $x \in \sum^*$. Table entries as illustrated in Figure are filled as follows; if you find x_i in A_j, place 1 in the corresponding cell, else place 0.
We now show that there is a subset in $\sum^*$ which is not listed as part of the enumeration.

	x1	x2	x3	x4 ...
A1	1	0	1	0...
A2	0	1	1	0...
A3	1	1	0	1...
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Fig. 5: Listing $P(\sum^*)$ and $\sum^*$ in a Matrix

Choose $B = \{x_j \mid (A_j, x_j) = 0\}$. That is, include x_j in B if x_j is not in A_j. Clearly, $B \in P(\sum^*)$ but not listed as part of the enumeration. B differs from each A_i in at least one element (diagonal element). Therefore, the enumeration is incomplete. Thus our assumption is wrong. Hence, $P(\sum^*)$ is uncountable.

Problem 15: How many computational problems are there? Is it countable/uncountable.

Solution: 1
Let problem-1 asks for printing $\{0\}$, problem-2 asks for printing $\{0, 1\}$, ..., problem-i asks for printing set containing i elements from N. i.e., each problem prints a subset of N. This problem collection is same as counting i element subset of $P(N)$, power set of N. Clearly, all are computational problems (well defined input and output) and hence the number of computational problems is strictly greater than $\mid P(N) \mid$. We have already shown that $P(N)$ is uncountable, therefore, the number of computational problems is uncountable.

Solution:2
In the earlier section, while showing the number of C-programs is countably infinite, we assumed that the alphabet is finite alphabet containing all ASCII characters. This assumption is true while counting the number of programs as program size is finite. However, this

assumption need not be true for describing the problem. In other words, consider a computational problem, `Print the irrational number 0.524673...`, this description contains an infinite string as a substring. So, the problem description takes infinite characters, which is a $P(\sum^*)$. Thus, the number of computational problems is uncountable.

Remark on Solvability/Unsolvability:
Every problem is either solvable or unsolvable. A problem is said to be solvable if there exists an algorithm (or a program that terminates with well defined input and output). A problem is said to be unsolvable if there does not exists an algorithm (Example: Print N, Print I).

Problem 16: How many solvable problems are there ?
Since each solvable problem has an algorithm, this count is equivalent to the number of C-programs that halt with well defined input and output. Therefore, the number of solvable problems is countably infinite.
Problem 17: How many unsolvable problems are there ?
Consider the set of programs `print` $[i, j]$ `where` $i, j \in R$, each program in this set is an unsolvable problem as there is no algorithm to list all values of the closed interval $[0, 1]$ and this is true for all $[i, j]$. Since the size of this set is R, the number of unsolvable problems is uncountable.

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