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COM205T Discrete Structures for Computing-Lecture Notes

## Graph Theory

### Motivation

1. Given a road network, find a minimum number of policemen so that every road is monitored. (policemen are placed at junctions, and in case of accidents at road  $r$  it will be addressed by the policeman standing at junction on any one end of  $r$ )
2. Design a router network so that it can handle all 2-node failures. (Fault tolerance level is 2)
3. Consider the interaction between processor and resources. Design an inter-process resource network so that there are no cyclic interactions (deadlock)

### Graphs

- An abstract representation of a system under study (system: computer network, road network, router network)
- used as a model to understand the system better
- it is a binary relation
- graphs consist of vertices (nodes) and edges (links/arcs)

### Basic Definitions and Simple Counting

$V(G) = \{v_1, v_2, \dots, v_n\}$ , the set of vertices.

$E(G) \subseteq V(G) \times V(G)$ , the set of edges, also represents the binary relation.

Ex:  $V(G) = \{1, 2, 3, 4\}$   $E(G) = \{(1, 1), (1, 3), (3, 4)\}$

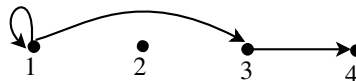


Fig. 1. A Graph

The graph in example is an example of a directed graph. If  $E(G) = \{\{u, v\} \mid u \text{ is adjacent to } v\}$ , then the underlying graph is an undirected graph. Undirected graph is a special case of directed graph where in the edge  $\{u, v\}$  in undirected graph represents the edges  $(u, v)$  and  $(v, u)$  in the corresponding directed graph.

1. How many different directed graphs on  $n$ -vertices are possible?

Since each directed graph corresponds to a binary relation,

The number of  $n$ -vertex graphs = the number of binary relations possible on a set of size  $n$  is  $2^{n^2}$

**Definition** *Simple graphs* are graphs with no self loops and no multiple edges.

Ex:  $V(G) = \{1, 2, 3, 4\}$   $E(G) = \{(1, 3), (3, 4)\}$ , is an example simple graph. The graph given above is not a simple directed graph.



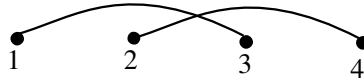
**Fig. 2.** A Simple Directed Graph

2. How many directed simple graphs are there on  $n$ -vertices?

Ans. The number of such graphs are equivalent to the number of irreflexive relations on a set of size  $n$  is  $2^{n^2-n}$

From now on, we shall work with simple undirected graphs.

Ex:  $V(G) = \{1, 2, 3, 4\}$   $E(G) = \{\{1, 3\}, \{2, 4\}\}$



**Fig. 3.** An Undirected Graph

3. How many different undirected simple graphs are there on  $n$ -vertices?

Ans. The number of such graphs are equivalent to the number of irreflexive and symmetric relations on a set of size  $n$  is  $2^{\binom{n}{2}}$

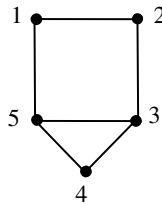
4. How many undirected simple graphs are there on  $n$  vertices and  $l$  edges.

Ans. Total number of edges possible is  $\binom{n}{2}$  and any subset of size  $l$  from  $\binom{n}{2}$  is an example graph on  $n$  vertices and  $l$  edges. Therefore, the number of graphs on  $l$  edges is  $\binom{\binom{n}{2}}{l}$ . ( $n$  choose 2, choose  $l$ )

### Undirected simple graphs and some more definitions

For a graph  $G$ , the **neighborhood** of a vertex  $v$  is  $N_G(v) = \{u \mid \{u, v\} \in E(G)\}$ .

Eg:  $N_G(3) = \{2, 4, 5\}$ ,  $N_G(4) = \{3, 5\}$



The **degree** of a vertex  $d_G(v)$  is the number of edges incident on  $v$ .  $d_G(v) = |N_G(v)|$

The degree sequence of  $G$  is denoted as  $(d_1, d_2, \dots, d_n)$ , where  $d_i$  is the degree of vertex  $v_i \in V(G)$ .

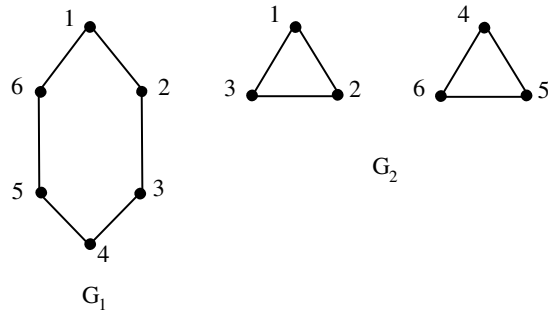
**Example:** Consider the above graph  $G$ , with  $d_G(1) = 2, d_G(2) = 2, d_G(3) = 3, d_G(4) = 2, d_G(5) = 3$ . The degree sequence is  $(3, 3, 2, 2, 2)$ .

**Note:** Given a degree sequence, one can construct the associated graph in more than one way. For the degree sequence  $(2, 2, 2, 2, 2, 2)$ , the two associated graphs are given below;

**Definition: Connectedness** A graph  $G$  is connected if for every  $u, v \in V(G)$  there exists a path between  $u$  and  $v$

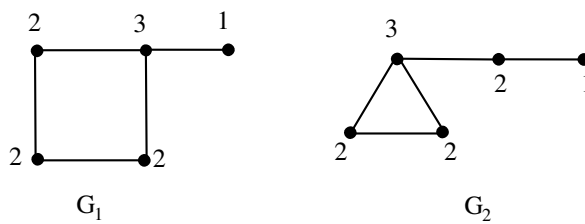
In the above figure,  $G_1$  is connected whereas  $G_2$  is disconnected with two components.

**Connected component** is a maximal connected subgraph of a graph. Note that maximal is with respect to a property, and here it is connectedness.



We see a natural extension of the previous question as follows. Given a degree sequence  $(d_1, d_2, \dots, d_n)$  can you construct the associated connected graph uniquely?

Ans. No. Consider the degree sequence  $(3, 2, 2, 2, 1)$ , there are two associated graphs as shown below;



**Fig. 4.** Two non isomorphic representation of  $(3, 2, 2, 2, 1)$

Question: Given two graphs  $G_1$  and  $G_2$ , how can you determine that they are different?

### Isomorphism

Two graphs  $G$  and  $H$  are isomorphic if and only if there exists a bijection from  $V(G)$  to  $V(H)$ .  $f : V(G) \rightarrow V(H)$  such that  $\{u, v\} \in E(G)$  if and only if  $\{f(u), f(v)\} \in E(H)$ .

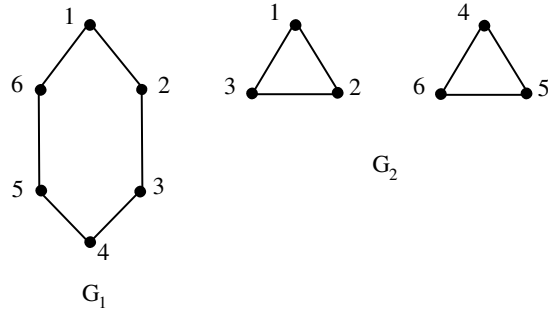
In Figure 4, there does not exist such a bijection from  $V(G) \rightarrow V(H)$ . Isomorphism highlights structural similarity between two graphs.

### Remarks:

1. A connected graph and a disconnected graph cannot be isomorphic to each other.
2. A graph containing a cycle and an acyclic graph cannot be isomorphic to each other.
3. There may be many bijections between  $V(G)$  and  $V(H)$ , we are interested in the one that preserves edge adjacency and non adjacency.
4. If two graphs are isomorphic then the number of vertices and the number of edges of those two graphs are same. The converse is false as illustrated below.

### Questions

1. Given  $(d_1, d_2, \dots, d_n)$ , how will you construct  $G$ .
2. Given  $G$  and  $H$ , how do you check whether they are isomorphic or not? Also, produce the associated bijection, if it exists.
3. In a group of  $n$  people, how many handshakes are possible?- Ans:  $\binom{n}{2}$
4. Are there graphs with the degree sequence
  - (i)  $(3, 3, 3, 3, 3)$  (ii)  $(3, 3, 3, 4, 4, 2)$  (iii)  $(1, 2, 2, 2)$  (iv)  $(5, 4, 3, 2, 1)$



**Remarks:**

1. For any graph, the maximum degree is at most  $n - 1$ . If a degree sequence of size  $n$  contains a vertex whose degree is more than  $n$ , then there is no graph corresponding to the degree sequence.
2. For any graph, we cannot have two vertices such that one is of size 0 and the other is of size  $n - 1$ .
3. The above two conditions are necessary but not sufficient. That is, there are many sequences that satisfy the above two conditions and not graphic.

**Some Structural observations on Graphs**

Claim 1:  $\sum_{i=1}^n d_i = \text{Even}$

Claim 2:  $\sum_{i=1}^n d_i = 2|E|$

Induction on  $m = |E(G)|$

*Proof. Base case:*  $m = 1, \sum_{i=1}^n d_i = 2$  is even

*Induction Hypothesis:* Assume that the claim is true for graphs with less than  $m$  edges,  $m \geq 2$ .

*Induction Step:* Consider the graph with  $m$ -edges;  $m \geq 2$  and let  $\{u, v\} \in E(G)$ .

Consider the graph  $G - \{u, v\}$ .  $V(G - \{u, v\}) = V(G)$  and  $E(G - \{u, v\}) = E(G) \setminus \{u, v\}$ .

Since  $|E(G - \{u, v\})| = m - 1$ , we can bring in the induction hypothesis.

By the Induction hypothesis, in  $G - \{u, v\}$ ,  $\sum_{i=1}^n d_i = 2m' = 2(m - 1)$ .

Add  $\{u, v\}$  to  $G - \{u, v\}$ . Consider the degree sequence  $d_1 + d_2 + \dots + d_u + d_v + \dots + d_n$ .

$d_u = d_{u'} + 1, d_v = d_{v'} + 1, u'$  and  $v'$  are the vertices corresponding to  $u$  and  $v$  in  $G - \{u, v\}$ . By introducing the edge  $\{u, v\}$ , the degree of  $u'$  ( $v'$ ) increases by one.

$$d_1 + d_2 + \dots + d_{u'} + 1 + d_{v'} + 1 + \dots + d_n.$$

$$= d_1 + d_2 + \dots + d_{u'} + d_{v'} + \dots + d_n + 2 = 2(m - 1) + 2 = 2m. \quad \square$$

We shall next present another inductive proof of the above claim; induction on  $|V(G)|$

*Proof. Base case:*  $n = 1, \sum_{i=1}^n d_i = 0$  is even

*Induction hypothesis:* Assume the claim is true for graphs with  $(n - 1)$ -vertices,  $n \geq 2$ .

*Induction Step:* Let  $G$  be a graph on  $n$ -vertices  $n \geq 2$

$$V(G) = \{u_1, u_2, u_3, \dots, u_n\}$$

Let  $u_i$  be a vertex with minimum degree ( $\delta(G)$ )

Consider the graph  $G - u_i$ .  $|V(G - \{u_i\})| = n - 1$  and  $|E(G - \{u_i\})| = |E(G)| - d_{u_i}$

By the induction hypothesis, the claim is true in  $G - u_i$

i.e.,  $d_{u_1} + d_{u_2} + \dots + d_{u_{i-1}} + d_{u_{i+1}} + \dots + d_{u_n} = 2m'$   
 $d_{u_1} + d_{u_2} + \dots + d_{u_{i-1}} + d_{u_{i+1}} + \dots + d_{u_n} = 2(m - \delta(G))$ , due to the removal  $\delta(G)$  edges incident on the minimum degree vertex.

By introducing  $u_i$  in  $G - u_i$ , we can see that every vertex  $v \in N_G(u_i)$ ,  $d_G(v)$  is increased by one. Now,  $d_{u_1} + d_{u_2} + \dots + d_{v_1} + d_{v_2} + \dots + d_{v_{\delta(G)}} + \dots + d_{u_n} + d_{u_i}$  where  $\{v_1, v_2, \dots, v_{\delta(G)}\} = N_G(u_i)$ . Also, note that  $d_{u_i}$  is added to the sum as  $u_i$  is added.

$$\begin{aligned} & d_{u_1} + d_{u_2} + \dots + d_{v'_1} + 1 + d_{v'_2} + 1 + \dots + d_{v'_{\delta(G)}} + 1 + \dots + d_{u_n} + d_{u_i} \\ &= d_{u_1} + d_{u_2} + \dots + d_{v'_1} + d_{v'_2} + \dots + d_{\delta'(G)} + \dots + d_{u_n} + 1 + 1 \dots + 1 + d_{u_i} \text{ [ no.of 1's}=\delta(G) \text{ and} \\ & d_{u_i} = \delta(G) \text{ ]} \end{aligned}$$

By I.H.  $\implies 2(m - \delta(G)) + \delta(G) + \delta(G) = 2m$ . This completes the proof.  $\square$

Based on the above claim, here is an interesting corollary; Let  $V_{\text{ODD}} = \{u \mid d_G(u) : 2k + 1, k \geq 0\}$  and  $V_{\text{EVEN}} = \{u \mid d_G(u) : 2k, k \geq 0\}$

$$\sum_{u \in V_{\text{ODD}}} + \sum_{u \in V_{\text{EVEN}}} = 2m$$

implies  $\sum_{u \in V_{\text{ODD}}}$  is even

**Claim 3:** The number of odd degree vertices in any graph is always even.  
 Corollary of claim 2.

### Some Special Graphs

*Path graphs* A path graph  $P_n$  on  $n$  vertices is defined as follows;  $V(P_n) = \{v_1, \dots, v_n\}$ ,  $E(P_n) = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq (n - 1)\}$ .

Note  $|V(P_n)| = n$ ,  $|E(P_n)| = n - 1$

*Cycle graphs* A cycle graph  $C_n$  on  $n$  vertices is defined as follows;  $V(C_n) = \{v_1, \dots, v_n\}$ ,  $E(C_n) = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq (n - 1)\} \cup \{v_1, v_n\}$ .

Note  $|V(C_n)| = n$ ,  $|E(C_n)| = n$

For a graph, the number of edges is in the range  $[0, \binom{n}{2}]$ . Graphs with  $\binom{n}{2}$  edges are called complete graphs.

*Complete graphs* A complete graph  $K_n$  on  $n$  vertices is defined as follows;  $V(K_n) = \{v_1, \dots, v_n\}$ ,  $E(K_n) = \{\{v_i, v_j\} \mid v_i, v_j \in V(K_n)\}$ .

*Regular graphs:*  $G$  is  $k$ -regular if for every  $v \in V(G)$ ,  $d_G(v) = k$

Ex:  $C_n$  is 2-regular,  $K_n$  is  $(n - 1)$ -regular.

The number of edges in a  $k$  regular graph on  $n$  vertices  $= \frac{nk}{2}$

Are there 3-regular graph on 7 vertices - No

*Trees:* A tree is a connected acyclic graph. A tree on  $n$  vertices has  $n - 1$  edges.

### Bipartite graphs:

$G$  is a bipartite graph if there exists a partition  $V_1, V_2$  of  $V(G)$  such that  $V(G) = V_1 \cup V_2$  and  $V_1 \cap V_2 = \emptyset$ . For every edge  $e = \{u, v\} \in E(G)$ ,  $u \in V_1$  and  $v \in V_2$

Example bipartite graphs include  $P_n$ ,  $C_{2n}$ , and all trees.  $K_n, n \geq 3$  and  $C_{2n+1}, n \geq 1$  are not bipartite.

**Question:** Does there exist a characterization for a graph to be bipartite ?

**Claim 4:**  $G$  is bipartite if and only if  $G$  is odd-cycle free.

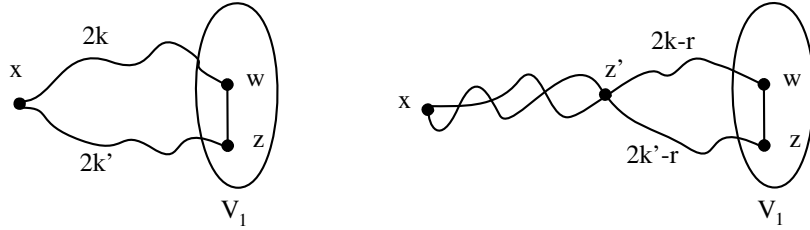
*Proof. Necessity:* Since  $G$  is bipartite, there exist  $V_1, V_2$  such that  $V(G) = V_1 \cup V_2$ ,  $V_1 \cap V_2 = \emptyset$ , and for every edge  $e = \{u, v\} \in E(G)$ ,  $u \in V_1$  and  $v \in V_2$

Consider  $u \in V_1$  and a cycle  $C$  starting and ending at  $u$ .

Since any cycle  $C$  that starts and ends at  $u$  visits vertices of  $V_1$  and  $V_2$  alternately, the length of  $C$  is clearly even.

*Note:* for any  $\{x, y\} \subseteq V_1$ , distance between  $x$  and  $y$  is  $2k + 1$ ,  $k \geq 1$ . Therefore, the length of cycle  $C$  is  $2k + 1 + 1$ , which is even.

*Sufficiency:*  $G$  is odd cycle free. To show that  $G$  is 2-partite, we need to exhibit a bipartition.



**Fig. 5.** An illustration for the proof of Claim 3

Let  $x$  be a vertex in  $G$ .

Consider  $V_1 = \{u \mid \text{distance}(x, u) = \text{even}\}$

$V_2 = \{u \mid \text{distance}(x, u) = \text{odd}\}$

Claim 1:  $V = V_1 \cup V_2$

Claim 2:  $V_1 \cap V_2 = \emptyset$

Claim 3: For each  $w, z \in V_1$ ,  $\{w, z\} \notin E(G)$ .

*Proof:* Suppose  $\{w, z\} \in E(G)$ , then  $|P_{xw}| = 2k$  and  $|P_{zx}| = 2k'$  for some  $k', k \geq 1$  as shown in figure.  $(P_{xw}, \{w, z\}, P_{zx})$  is a cycle of length  $2k + 2k' - 1 = 2l + 1$  for some  $l \geq 1$ . Therefore,  $G$  contains an odd cycle and this is a contradiction to the premise. Our assumption that there exists  $\{w, z\} \in E(G)$  is wrong and  $\{w, z\} \notin E(G)$ . Therefore,  $V_1$  is an independent set. Similar arguments hold true if  $V_2$  is an independent set.

Suppose,  $V(P_{xw}) \cap V(P_{zx}) \neq \emptyset$ , then identify the last vertex  $z'$  such that  $z' \in P_{xw}$  and  $z' \in P_{zx}$ . Let the length of  $P_{xz'}$ ,  $|P_{xz'}| = r$ . Note that  $P_{xz'} \subseteq P_{xw}$  and  $P_{xz'} \subseteq P_{zx}$  are of length  $r$ . Suppose  $|P_{xz'}| < r$ , then it contradicts the fact that  $P_{xz}$  is a shortest path. It follows that  $|P_{zz'}| = 2k' - r$  and  $|P_{z'w}| = 2k - r$ . The length of cycle  $(P_{z'w}, \{w, z\}, P_{zz'})$  is  $2k - r + 2k' - r - 1 = 2l + 1$  for some  $l \geq 1$ . Therefore, there exists an odd cycle which is a contradiction. Therefore, the assumption is wrong, and the claim follows.  $\square$

**Questions:**

A graph is 3-partite if and only if ---? Can one come up with a forbidden structure similar to bipartite graphs

What about a necessary and sufficient condition for a graph to be  $k$ -partite ?

**Subgraphs**

A graph  $H$  is a subgraph of  $G$  if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ .  $H$  is an induced subgraph of  $G$ , if  $H$  is a subgraph of  $G$  and  $\{u, v\} \in E(G)$  if and only if  $\{u, v\} \in E(H)$ . For example,

1.  $P_n$  is a subgraph of  $C_n$  but not induced.  $P_{n-1}$  is an induced subgraph of  $C_n$ . For  $1 \leq k \leq n-1$ ,  $P_k$  is an induced subgraph of  $C_n$ .
2.  $C_n$  is a subgraph of  $K_n$  but not induced,  $n \geq 4$ .  $K_{n-1}$  is an induced subgraph of  $K_n$ .
3. Any  $K_n$  contains a  $k$ -regular induced subgraph,  $1 \leq k \leq (n-1)$ .
4. Any connected graph on  $n$  vertices contains a tree on  $n$  vertices as its subgraph.
5.  $K_n$  contains every cycle of length  $k$ ,  $3 \leq k \leq n$ .

### Induced Cycle

A chord in a cycle is an edge connecting non consecutive vertices in a cycle. A cycle without chords is an induced cycle. Any  $C_n$ , by definition, induced.  $K_4$  has induced  $C_3$  but not induced  $C_4$ ; contains  $C_4$  but not induced  $C_4$ . Any  $K_n$ ,  $n \geq 3$ , contains induced  $C_3$  but no induced  $C_k$ ,  $k \geq 4$ .

### Hamiltonian and Eulerian graphs

The Hamiltonian cycle (path) is spanning cycle (path) containing each vertex exactly once. A connected graph is Hamiltonian (cycle or path) if it contains Hamiltonian cycle (path). The Eulerian cycle (path) is a spanning cycle (path) containing each edge exactly once. A vertex may appear more than once. A connected graph is Eulerian if it contains Eulerian cycle (path). Eulerian cycle are also referred to as Eulerian circuit in the literature.

1.  $P_n$  has Hamiltonian path but not Hamiltonian cycle. Similarly,  $P_n$  has Eulerian path but not Eulerian cycle.
2.  $K_n$  has Hamiltonian cycle and Hamiltonian path. If a graph has Hamiltonian cycle then it has Hamiltonian path as well. The converse need not be true.
3.  $K_{2n+1}$  has Eulerian cycle and path.

### Self Complementary graphs

For a graph  $G$ , the complement of  $G$  is denoted as  $\bar{G}$  which is,  $V(\bar{G}) = V(G)$ ,  $E(\bar{G}) = \{\{u, v\} \mid \{u, v\} \notin E(G)\}$ . If a graph  $G$  is isomorphic to  $\bar{G}$  (complement of  $G$ ), then  $G$  and  $\bar{G}$  are self complementary graphs. The complement of  $P_4$  is  $P_4$  itself, and therefore  $P_4$  and its complement are self complementary graphs.  $C_5$  and its complement which is  $C_5$  itself, are self complementary graphs.

**Line graphs** For a graph  $G$ , the line graph  $L(G)$  is defined as  $V(L(G)) = \{e \mid e \in E(G)\}$ ,  $E(L(G)) = \{\{e, e'\} \mid e \text{ is adjacent to } e' \text{ in } G\}$ . The line graph of  $P_n$  is  $P_{n-1}$ . The line graph of  $C_n$  is  $C_n$ . The line graph of  $K_4$  is a 4-regular graph on 6 vertices.

### Planar Drawing and Planar Graphs

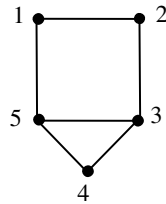
A plane drawing is a drawing of edges in which no two edges cross each other. A graph is a planar graph if there exists a plane drawing.  $K_4$ , any  $P_n$ , any  $C_n$ , any tree are planar graphs.  $K_5$  and  $K_{3,3}$  (complete bipartite graph with partition size is 3 each) are non-planar. Interestingly,  $K_5 - e$  and  $K_{3,3} - e$  (exactly one edge is removed from  $K_{3,3}$ ) are planar.

1. If a subgraph is non-planar then the graph (super graph) is non-planar.
2. If  $G$  is planar then the number of edges is,  $|E(G)| \leq 3|V(G)| - 6$ . That is,  $m \leq 3n - 6$ .
3. The minimum degree in any planar graph is at most 5. If suppose, minimum degree is at least 6, then the degree sum is  $6n$  and the number of edges is at least  $3n$ . A contradiction to the previous result.
4. Any closed region is referred to as **face** in a graph (also known as interior face). The exterior face of a planar graph refers to the plane on which the graph is drawn. For acyclic graphs, there is no interior face and has exactly one exterior face.

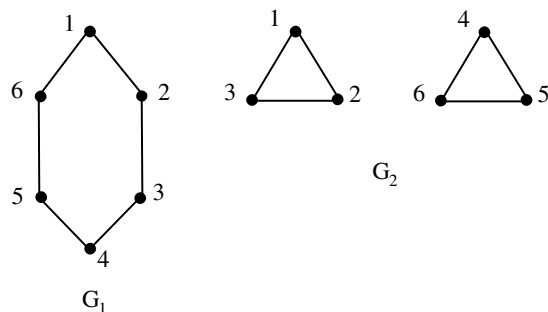
For  $C_n$ , the number of faces is two; for trees, the number of faces is one. The following formula due to Euler relates  $n, m$  and the number of faces ( $f$ ). For any connected planar graph,  $n - m + f = 2$ . For a disconnected graph, the above formula is applicable for each connected component.

5. For the following graph,  $n = 5, m = 6, f = 3, n - m + f = 2$  is verified.

6. The following graph is disconnected and hence, Euler's formula is applied on each connected



component. For component one,  $6 - 6 + 2 = 2$ . For the second and third components,  $3 - 3 + 2 = 2$ .



7. For an edge  $e = \{u, v\}$ , the contraction of  $e$  is defined as follows;  $V(G \cdot e) = (V(G) \setminus \{u, v\}) \cup \{z\}$ ,  $E(G \cdot e) = (E(G) \setminus (\{\{u, x\} \mid x \in V(G)\} \cup \{\{v, x\} \mid x \in V(G)\})) \cup \{\{z, x\} \mid x \in N_G(u) \cup N_G(v)\}$ . That is, contraction of an edge  $e = \{u, v\}$  removes the end points  $u$  and  $v$ , and introduces a new vertex  $z$  such that  $z$  is adjacent to  $N_G(u) \cup N_G(v)$ .

8. The graph  $G \cdot e$  is a minor of  $G$ . If  $K_5$  or  $K_{3,3}$  can be obtained through a sequence of edge contractions, then the underlying graph is non-planar.

9. Kuratowski's Result:  $G$  is planar iff  $G$  does not have  $K_5$  or  $K_{3,3}$  minor.

## Graph Coloring

- An assignment of colors to vertices of a graph
- *Proper coloring* - adjacent vertices receive different colors.
- $G$  is  $k$ -colorable if and only if there exists  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  such that for all  $e = \{u, v\}, c(u) \neq c(v)$
- *Chromatic Number*  $\chi(G)$  is the minimum number of colors required to properly color a graph.

$$\chi(K_n) = n$$

$$\chi(P_n) = 2$$

$$\chi(Tree) = 2$$

$$\chi(C_{2n}) = 2$$

$$\chi(C_{2n+1}) = 3$$

$$\chi(bipartite\ graph) = 2$$



$G$  is bipartite if and only if  $G$  is 2-colorable.

The following statements are equivalent.

- $G$  is bipartite
- $G$  is 2-colorable
- $G$  is odd-cycle free

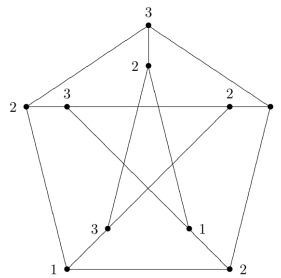
### Clique, Independent Set and Matching

**Clique:** A set  $S \subseteq V(G)$  is a clique, if  $\forall u, v \in S, \{u, v\} \in E(G)$ . That is, a fully connected subgraph of  $G$ . Any  $K_n$  is a clique. Any tree contains a clique of size 2 (any edge) and no cliques of size at least 3. For all bipartite graphs, maximum clique size is 2.

**Independent set:** A set  $S \subseteq V(G)$  is an independent set, if  $\forall u, v \in S, \{u, v\} \notin E(G)$ . In  $P_n$ , the alternate vertices form an independent set. The size of independent set in  $P_n$  is  $\frac{n}{2}$ . In a tree, the set of leaves is an example independent set.

**Matching:** A set  $E' \subseteq E(G)$  is a matching, if  $\forall e, e' \in E(G), e \cap e' = \phi$ . That is a set of edges such that no two edges share a vertex in common. A perfect matching is a matching that covers each vertex.  $P_4$  has a perfect matching, whereas  $P_5$  does not have a perfect matching.

**The Peterson Graph:** The Peterson graph, named after the famous mathematician Peterson, is given below;



**Fig. 6.** The Peterson graph

The Peterson graph has many interesting properties (i) 3-regular (ii) 3-vertex colorable and 3-edge colorable (iii) non-bipartite (iv) non-planar (v) Has Hamiltonian path but not Hamiltonian cycle (vi) Non-Eulerian (v) Has induced  $C_5, C_6$ . To prove non-planarity, contract all edges connecting outer  $C_5$  and inner star on 5 vertices, the resultant is  $K_5$ .

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1. K.H.Rosen, Discrete Mathematics and its Applications, McGraw Hill, 6th Edition, 2007
2. D.F.Stanat and D.F.McAllister, Discrete Mathematics in Computer Science, Prentice Hall, 1977.
3. C.L.Liu, Elements of Discrete Mathematics, Tata McGraw Hill, 1995

## Reading assignment

[1]. Shakuntala Devi: "Puzzles to Puzzle you"

"More Puzzles",

"Figuring: The Joy Of Numbers"

[2]. George J. Summers: "The Great Book of Mind Teasers and Mind Puzzlers"

### A MATTER OF TIME [1]

*Fifty minutes ago if it was four times  
as many minutes past three o'clock.*

*How many minutes is it to six o'clock*