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COM205T Discrete Structures for Computing-Lecture Notes

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`Introduction to Discrete Mathematics and Logic`

Objective of this course:

- To have an understanding of the principles of logic.
- To get a flavor of the art of writing proofs.
- To learn to reason and analyze the validity of statements in a given context.
- To get familiar with enumeration and counting problems.
- To have a tour on infinite sets to know about countable and uncountable sets.
- To study the discrete object 'Graphs' and its modeling power in computational problems.

We shall begin this lecture with a set of fundamental questions. And every attempt to answer these questions will take us a long way in understanding this subject better. A deeper understanding of these questions will give a good insight into the relatively young science, namely, computer science.

Some Interesting Questions to Ponder On:

1. How was computer science born?
2. Is there a science in computer science?
3. Like human beings, can a machine think, cook, sing, and drive

Let us first understand, how other fields of science and engineering were discovered over time. A look into this evolution may yield us with some fruitful insights which will help us in answering the above questions. We shall also introduce Logic and its power in expressing facts with no ambiguity.

1 Discoveries (inventions) at a Glance

Zeroth machine age:
This was the time up to early nineteenth century, where there were no noticeable developments in the technology. Preaching and teaching was followed in domestic languages, and knowledge dissemination was not quite high. Some foreign invasions and battles among kingdoms were popular then. Over the years, people became more closer and communications were strengthened. This helped in exchange of knowledge, even though language was a barrier.
Aristotle, a Greek philosopher, who lived in fourth century BC, was one of the pioneers who paved the basics of Logic. He observed general laws of nature which were not immediately perceivable to a common man. Observations from nature helped him to draw probable conclusions, sometimes with undeviating precisions. Perception of physical phenomenon, their causes and their evidence are not derived mathematically. Instead they were outcomes of extensive observations, reasoning and numerous experience of practical verification. People who do these types of

researches were treated as philosophers in the ancient times. Aristotle was the odd man out, as he not only was interested in giving claims, but also in giving sufficient reasoning. He felt that the observations and conclusions when represented in any natural language will be ambiguous and hence the principles and theorems may not be interpreted in its pure form. Therefore, his investigations were concentrated on the reasoning and representation of principles and theorems (along with the proof), in scientific form. An understanding between the reader and presenter, on the general reasoning and operations presented in the proof, is necessary for its unambiguous reception. Towards this objective, he devised symbols needed for proving claims, mentioned the precise meaning of the symbols and arguments used, and presented the proof. This age is known as zeroth machine age.
In zeroth machine age, knowledge transfer happened among individuals. Precise delivering of knowledge in a human-human interaction required scientific presentation, which was the initial motivation for the study of logic.

First machine age:

Starting from the inventions like nuts, bolts and wheels, this age had witnessed major inventions and discoveries that changed every facet of life. It was an age of globalization. And it was in this age that the major three engineering areas (Electrical, Civil and Mechanical) flourished. Rise of machines necessitated human-machine interaction, which required more precise representation of facts.

After a long gap, in nineteenth century, an English Mathematician and Philosopher George Boole made further research in logic methodologies and took the lead in popularising this area, which was even taught as a subject in Harvard University after his death. He contributed remarkable developments and theories in logical research, that modern logicians are indeed to be inevitably thankful for. His masterpiece includes *Mathematical Analysis of Logic*, *The Laws of Thought*, where he shows that the science of logic is powerful as it is extensively researched upon even today.

Claude Shannon (1916-2001)

"I visualise a time when we will be to robots what dogs are to humans, and I'm rooting for the machines."

Second machine age:

Middle of twentieth century can be considered as the golden time of research, as modern technologies started evolving at this time, where research in atomic particles extended research deep into new specialized areas in technology which included electronics, and computer engineering. Unlike other machines with which humans interacted, computers require extremely precise information. Computers have been used for doing scientific calculations and is considered even now as the greatest technological leap of mankind. Machine-machine interactions observed in this age could not have occurred without the invention of logic gates. As life progresses, automation of each and every processes that we have come across in daily life became the need of the hour. For this purpose, mapping of inputs to logical objects became necessary, so that a practical problem in our daily life is mapped into mathematical domain and is solved using computers. Thus, `Logic` as a language to program computers evolved during this time.

2 Introduction to Propositional Logic

Having given motivation for logic in the previous section, we shall now present logic in detail.

`Science in Computer Science:` A science about data: data representation, data storage and data processing. Also, a science that deals with problem solving using computers

Data representation focuses on the formal representation of data. The language logic offers a formal representation of data which is precise, concise and free from ambiguity. The formal representation helps in problem solving through computers. Further, logic provides quantification of data satisfying some predicate (property). Propositional logic (zeroth order logic) is a simple language that offers no quantification of data, whereas predicate logic (first order logic) offers quantification of data.

Do you know ?

Coloring of regions in a map on a plane such that adjacent regions sharing boundaries (not points) receive different colors can be done with Four *colors !!!*

Terminologies:

An `Assertion` is a meaningful statement. `Proposition` is an assertion which is either True or False, but not both. Further, a `proposition` is a declarative statement which is either true or false, but not both. For example; (a) $2 + 3 = 5$, is a declarative statement which is true always and hence, it is a proposition. (b) The product of LCM and GCD of two numbers is precisely the product of those two numbers, is a declarative statement which is true always, and hence it is a proposition. (c) Every even number ends with a digit 1 or 3 is a statement which is false always, and hence it is a proposition. (d) This day is Friday, is a declarative statement which takes both true and false depending upon the day. If it is a Friday, then the statement is true, and false otherwise. Hence, (d) is not a proposition. (e) Statements such as 'get me a pen drive (imperative)', 'great! you did it (exclamatory)', 'Is there a science in computer science ? (interrogative)', are not declarative sentences. (f) Discrete mathematics is a fascinating subject is a statement which takes both true and false depending upon the set of students under consideration, and hence not a proposition.

Note that the language logic focuses on the truth value of the proposition, not just the statement itself. We introduce the first level abstraction by introducing propositional variables. Propositional variables represent propositions, and propositional formula consists of propositional variables and logical operations relating them. Essentially, simple declarative statements which are propositions, are represented using propositional variables and complex declarative statements (statements that can be split into a collection of simple statements) which are propositions, can be represented using propositional formula using appropriate logical operations. The truth value of a propositional formula depends on truth values of individual propositional variables and the logical operators that operate on them. By assigning truth values to propositional variables, we add meaning to statements. Further, one can infer whether the formula is evaluated to true or false, always.

Propositions are denoted using *Proposition variables* P, Q, R, etc.

For example, P : Discrete mathematics is a classical subject in Computer Science

$Q : 2 + 7 = 9$.

If a proposition Q is true, then its *truth value* is True, denoted as truthvalue(Q)=True. Similarly for a False proposition R, truthvalue(R)=False. An *Axiom* is a proposition or statement which is regarded as being established, accepted or self-evidently True.

Theorem is a general proposition which is not self-evident but proved by a chain of reasoning; a truth accepted by means of accepted truths. A logical set of arguments which establishes the theorem to be `TRUE` is called *Proof*. *Lemma* is an intermediate theorem in an argument or proof.

Operations on Propositional Variables:
Usually we may handle more than one proposition for a proof which are joined using operators. For ease of representation, propositions are always represented using proposition variables. The commonly used operations on proposition variables are;

Consider the two propositional variables P : I study well,
Q : I excel in life.

- `Disjunction, also known as logical OR (inclusive)` The proposition $P \vee Q$ denotes the compound statement 'I study well or I excel in life' and $P \vee Q$ is TRUE if at least one of P, Q is true, and FALSE otherwise. This is 'inclusive or', it allows the possibility of both P and Q being true.
- `Logical OR (exclusive)` The proposition $P \oplus Q$ denotes the compound statement 'Either I study well or I excel in life' and $P \oplus Q$ is TRUE if exactly one of P, Q is true, and FALSE otherwise.
- `Conjunction, also known as logical AND` The proposition $P \wedge Q$ denotes the compound statement 'I study well and I excel in life' and $P \wedge Q$ is TRUE if both P, Q are true, and FALSE otherwise.
- `Negation` The proposition $\neg P$ denotes the statement; it is not the case that I study well. Equivalently, I do not study well.

OR,AND

P	Q	$P \wedge Q$	$P \vee Q$	$P \oplus Q$
T	T	T	T	F
T	F	F	T	T
F	T	F	T	T
F	F	F	F	F

NEGATION

P	$\neg P$
T	F
F	T

IMPLICATION (CONDITIONAL)

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

EQUIVALENCE (BICONDITIONAL)

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

The other operators which are commonly used in logical conversations are 'conditional (implication)' operator and 'biconditional (equivalence)' operator.

`Conditional Operator` $\to$:

The conditional statement $P \to Q$ is FALSE when P (premise) is true and Q is false (conclusion) and is TRUE, otherwise. We shall illustrate more on this through examples. Consider the two propositions;
P: It rains Q: I carry a rain coat.
Consider the statement "If it rains, then I carry a rain coat". Clearly, this is a proposition as the truth value (true or false) is determined by the truth values of premise and conclusion.
Let us analyse the truth value of this statement. (i) Suppose it rains and I carry a rain coat. Then, given proposition is respected and the truth value of the proposition is true. (ii) Suppose, it rains and I do not carry a rain coat. Then, proposition is not respected, the truth value of the proposition is false. (iii) Suppose, it does not rain. Note that, the proposition is taken for consideration if it rains. If it does not rain, then I may carry a rain coat or I may not. In this case, irrespective of the value of Q, the conditional statement is still true. Thus, the conditional statement is false when P is true and Q is false, and true, otherwise.

Consider the proposition made by your mathematics teacher, "If you solve all assignment questions, then you will get Grade 'S' in my course". Clearly, P: you solve all assignment questions. Q: you will get Grade 'S' in my course. (i) Suppose, you have solved all assignment questions and you are awarded 'S'. Then, the teacher kept his/her words and the proposition is respected, the truth value is true. (ii) Suppose, you have solved all assignment questions, but you are not awarded 'S'. Then, you are disappointed and the teacher did not keep his/her words. The truth value of the proposition is false. (iii) Suppose, you did not solve all assignment questions, then the proposition should not be taken into for consideration. The proposition is false, when the premise is true and the conclusion is false. When I did not solve all assignment questions, I may still get grade 'S' due to my performance in other examinations or I may not get grade 'S'.

Consider the statement; if a number is divisible by 10 then the number is divisible by 2. Clearly, if the premise is true, conclusion follows and hence the truth value of the expression is true. However, if a number is not divisible by 10, then the number may be divisible by 2, for example, the integer 4. In this case, the given conditional statement is still true. Further, we do not have a number which is divisible by 10 but not by 2. Note that in the last case, although there is no number which is divisible by 10 and not by 2, the given statement is still true.

When we look at propositional variables P, Q and expressions such as $P \rightarrow Q$, we do not give importance to instantiations of P and Q, and their relationships as the truth value associated with $P \rightarrow Q$ is important for discussion. For example; consider the proposition, if sun rises in the east then 2+3=5. The premise and conclusion have no relationships which we do not take into consideration as part of our discussion.

If sun rises in the west, then it is a sunset or sunrise. Observe that although, sun rising in the west will never happen, which means the premise is false always and irrespective of the conclusion, the truth value of the statement is true

Similarly, for the statement "If $2+3 = 6$ then $3 + 5 = 8$", the premise is false always, however, the truth value of the statement is true. Moreover, the conclusion is true always, therefore, the conditional statement is true.

If I excel in studies then 2+3=-4. Since the conclusion is false, the conditional statement is false.

If I study well then 2+3=5. Observe that the conclusion is true always and hence, irrespective of the truth value of the premise, the conditional statement is always true.

As far as truth table is concerned, if P and Q are arbitrary statements, then we should take all possible values of P and Q (4 rows in the truth table). However, if we instantiate P and Q with respect to a specific example, we need not consider all possible values. For example, for the statement if a number is divisible by 10 then the number is divisible by 2. The case when premise is true and conclusion is false need not be considered for discussion (only 3 rows for this example).

It is important to highlight that unlike natural language, logic adds meaning, helps to analyze and further, reason out each proposition. It is this meaning that helps to analyse whether a conversation is valid or invalid. There are many other expressions in English whose truth value is same as 'conditional expression' which we shall analyze now.

- if P, then Q

$-Q$ follows from P
$-Q$ is necessary for P
$-Q$ whenever P
$-P$ only if Q
$-P$ is sufficient for Q
$-Q$ if P
$-Q$ unless $\neg P$
$-Q$ provided P
$-Q$ is a consequence of P

`Q is a necessary condition (conclusion) for P:`. With respect to the example, "If you solve all assignment questions, then you will get Grade 'S' in my course", when some student solves all assignment questions, he is eligible for grade 'S'. There is no other requirement to get grade 'S'. To get 'S', one should solve all assignment questions and no other conditions. This means, Q is necessary whenever P is true.
`P is sufficient for Q:` To get grade 'S', it is enough to solve all assignment questions and no other conditions to be met. That is, when P is true, Q must follow. To conclude Q, it is enough to satisfy P.
`P only if Q:` This expression says, P cannot be true when Q is not true. The premise being true is influenced by the truth value of the conclusion. That is, if P is true and Q is false, then the truth value of P only if Q, is false. For example, consider the expression "Ice cream only if sugar". Any ice cream contains sugar and no ice cream can be prepared without sugar. Also means, if ice cream then it contains sugar for sure. Sugar follows from ice cream. Further, P only if Q means $\neg Q \to \neg P$ (when Q is not true, P is not true). Later, we shall see that $P \to Q$ and $\neg Q \to \neg P$ are equivalent through truth table method.

`Q unless ¬P:` This means, if $\neg P$ is false then Q is true. Consequently, one can write if $\neg P$ is false, then P is true and thus, if P is true, then Q is true. For example; consider the proposition, "I get grade 'S' unless I do not study". This means, if I study then I get grade 'S' which is precisely the conditional expression. Hence, the truth value of Q unless $\neg P$ and $P \to Q$ are same.
In natural language we use 'but', 'however', 'since', 'whereas', etc., and they all mean 'logical AND' in logic.

- I studied well, but I made a lot of mistakes; the equivalent statement in logic is I studied well and I made a lot of mistakes.
- I studied well, however I could not appear for the examination; in logic, it is expressed as I studied well and I could not appear for the examination
- Since it is five, we went out for playing; equivalent expression in logic is, it is five and we went out for playing.
- Discrete mathematics is a theory subject, whereas computer organization is a system subject; in logic, it is expressed as, Discrete mathematics is a theory subject and computer organization is a system subject.
- Though he was cool, he did well in all tests; in logic, he was cool and he did well in all tests
- Although we reached the examination hall on time, we were not allowed to write the examination; in logic, we reached the examination hall on time and we were not allowed to write the examinations.
- Despite having all credentials, he could not move up in his professional ladder; equivalent expression in logic, he was having all credentials and he could not move up in his professional ladder.

- In spite of her busy schedule, she gave the inaugural speech in a conference; in logic, she has busy schedule and she gave the speech in a conference.

`Biconditional operator` of two propositions P and Q, represented as $P \leftrightarrow Q$ is read as "P *if and only if* Q". $P \leftrightarrow Q$ is the conjunction of $P \rightarrow Q$ and $Q \rightarrow P$. Therefore $P \leftrightarrow Q$ has truth value TRUE when both P and Q has same truth values, and FALSE otherwise. The following statements are equivalent.

- P if and only if Q
- P is necessary and sufficient for Q
- if P, then Q and conversely if Q, then P.

Let R denote the proposition $P \rightarrow Q$. *Converse* of R is defined as $Q \rightarrow P$ and *Inverse* of R is defined by the implication $\neg P \rightarrow \neg Q$. *Contrapositive* of a proposition R is defined by the implication $\neg Q \rightarrow \neg P$. Note that the propositions constructed using the above mentioned operations are also propositions.

For the example, P : I study well, Q : I excel in life,
`Inverse:` If I do not study well, then I do not excel in life.
`Converse:` If I excel in life, then I study well
`Contrapositive:` If I do not excel in life then I do not study well

Two propositions are said to be logically equivalent if they have same truth table. Observe that $P \rightarrow Q$ is equivalent to $\neg Q \rightarrow \neg P$, represented as $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$. Similarly, $Q \rightarrow P \equiv \neg P \rightarrow \neg Q$. The above results may be verified using truth tables.

Compound proposition is a proposition formed from the propositional variables using the above defined operations. A *tautology* is a compound proposition whose truth value is TRUE for all values of its propositional variables occur in it. For example, $P \vee \neg P$ is a tautology. A *contradiction* or *absurdity* is a compound proposition whose truth value is FALSE for all values of its propositional variables occur in it. For example, $P \wedge \neg P$ is a contradiction. A *contingency* is a compound proposition which is neither a tautology nor a contradiction.

`Implication:` Given two logical expressions W and Z, To show that $W \rightarrow Z$ (W implies Z), we need to establish that $W \rightarrow Z$ is a tautology. For example, the following expressions are tautologies, and referred to as implications.

1. $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$.
2. $(P \rightarrow Q) \rightarrow (\neg P \vee Q)$

`Equivalence:` Given two logical expressions W and Z, To show that $W \leftrightarrow Z$ or $W \equiv Z$ (W is equivalent to Z), we need to establish that $W \leftrightarrow Z$ is a tautology. For example, the following expressions are tautologies, and are referred to as equivalences. To establish equivalences, we can also show that $W \rightarrow Z$ and $Z \rightarrow W$ are tautologies. Equivalence tells us that for every possible values of variables appearing in W and Z, the truth value of W and Z are same. The above implications are equivalences as well.

1. $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$.
2. $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
3. $(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow \neg Q)$

4. $(P \leftrightarrow Q) \leftrightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P))$
5. $(P \leftrightarrow Q) \leftrightarrow (\neg(P \oplus Q))$

`Remarks:`
1. Negation of a conditional statement: $\neg(P \rightarrow Q)$ is $\neg(\neg P \vee Q)$ which is $(P \wedge \neg Q)$. Thus, negation of "If I study well, then I get grade S" is "I study well and I do not get grade S".
2. Negation of a biconditional statement: $\neg(P \leftrightarrow Q)$ is $\neg(\neg(P \oplus Q))$ which is $(P \oplus Q)$. This also means that $\neg((P \rightarrow Q) \wedge (Q \rightarrow P))$, which is $(P \wedge \neg Q) \vee (\neg P \wedge Q)$.
3. $(P \oplus Q) \equiv (P \vee Q) \wedge \neg(P \wedge Q)$
4. Q is necessary for P but not sufficient: $(P \rightarrow Q) \wedge \neg(Q \rightarrow P)$.
5. Neither P nor Q: $(\neg P \wedge \neg Q)$.
6. Negation of "neither P nor Q" is not equivalent to "either P or Q". Negation of neither P nor Q is at least one of P, Q is true, which is $(P \vee Q)$. Negation of either P or Q is either both P and Q are true or both P and Q are false, which is $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
7. Increasing order of precedence: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$. For example; $\neg P \vee P \vee Q \wedge R \rightarrow Q \wedge S \leftrightarrow R$ is $((((\neg P) \vee P) \vee (Q \wedge R)) \rightarrow (Q \wedge S)) \leftrightarrow R$

Reading Assignment 1. Paradox 2. QED
3. Google Aristotle, George Boole, Alan Turing, De Morgan, R Smullyan, Alonzo Church 4. Articles from the text 1^{st}, 2^{nd} machine age, The Laws of Thought. 5. Do [1]Exercise 1.5

Logical representation and mechanism of logical reasoning acts as the basis of scientific investigation. It is to be noted that extensive and deep scientific research involves a large number of propositional variables. Compound propositions in these representations can be simplified using logical identities and implications. The table gives the well-known logical identities (equivalences) and implications, all can be verified using truth table method.

Is this right ?

Everyone loves my baby.
My baby only loves me.
$\therefore$ *I am my own baby.*

Check the validity of the following implications
$P \rightarrow (Q \rightarrow R)$ equivalent to $(P \rightarrow Q) \rightarrow (P \rightarrow R)$

P	Q	R	$Q \rightarrow R$	$(P \rightarrow Q)$	$(P \rightarrow R)$	$(P \rightarrow Q) \rightarrow (P \rightarrow R)$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	T	F	T	T	T
T	F	F	T	F	F	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence, $P \rightarrow (Q \rightarrow R)$ is equivalent to $(P \rightarrow Q) \rightarrow (P \rightarrow R)$.

[1] J.L.Mott, A.Kandel, and T.P.Baker: Discrete Mathematics for Computer Scientists and Mathematicians, PHI.

Equivalence		Name
$p \leftrightarrow (p \vee p)$	$p \leftrightarrow (p \wedge p)$	Idempotence
$p \vee q \leftrightarrow q \vee p$	$p \wedge q \leftrightarrow q \wedge p$	Commutativity
$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$	Associativity
$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$	$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$	De-morgans Law
$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$	Distribution
$p \vee \mathtt{T} \leftrightarrow \mathtt{T}$	$p \wedge \mathtt{F} \leftrightarrow \mathtt{F}$	Domination
$p \vee \neg p \leftrightarrow \mathtt{T}$	$p \wedge \neg p \leftrightarrow \mathtt{F}$	Negation
$p \vee \mathtt{F} \leftrightarrow p$	$p \wedge \mathtt{T} \leftrightarrow p$	Identity
$p \leftrightarrow \neg(\neg p)$		Double negation
$p \vee (p \wedge q) \leftrightarrow p$	$p \wedge (p \vee q) \leftrightarrow p$	Absorption law
$p \rightarrow q \leftrightarrow \neg p \vee q$		Implication
$(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$		Equivalence
$(p \wedge q) \rightarrow r \leftrightarrow p \rightarrow (q \rightarrow r)$		Exportation
$(p \rightarrow q) \wedge (p \rightarrow \neg q) \leftrightarrow \neg p$		Absurdity
$p \rightarrow q \leftrightarrow \neg q \rightarrow \neg p$		Contrapositive

Implication	Name
$p \rightarrow (p \vee q)$	Addition
$(p \wedge q) \rightarrow p$	Simplification
$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus Ponens
$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
$[\neg p \wedge (p \vee q)] \rightarrow q$	Disjunctive Syllogism
$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical Syllogism
$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	Resolution
$(p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)]$	
$[(p \rightarrow q) \wedge (r \rightarrow s)] \rightarrow [(p \wedge r) \rightarrow (q \wedge s)]$	
$[(p \leftrightarrow q) \wedge (q \leftrightarrow r)] \rightarrow p \leftrightarrow r$	

Questions:

− Verify using truth table. $((p \rightarrow q) \rightarrow r) \not\leftrightarrow (p \rightarrow (q \rightarrow r))$

− Prove $\neg(p \leftrightarrow q) \leftrightarrow (p \leftrightarrow \neg q)$

3 Predicate Logic

In the earlier section, we discussed propositional logic (zeroth order logic), its notation, compound expressions involving logical operators and connectives. In this section, we shall discuss predicate logic in detail.

As for formal representation of data, we need a notation which is abstract, precise, concise and yet close to the given data. Formalism should not add any additional meaning to or alter the meaning of the given data, however, in some cases, the language logic may not precisely express the context under consideration. This limitation is offered by the language itself as for example, expressions such as 'this boy might score well', 'this student with high probability score well', 'most of the students did well in the examination', 'his score is not that good', cannot be expressed precisely in logic. That is 'might', 'high probability', 'most', 'that good', etc., do not have a precise notation in logic. Moreover, logic cannot express feelings or emotions, implicitly or explicitly given in a statement. Despite these limitations, logic is powerful enough in expressing and proving many logical arguments. Consider the following examples;

- All boys are good.

- There are at least two students of this class eligible for B.Tech (Honours)
- Some students of this class like all forms of sports.

All of the above expressions are declarative statements taking the truth value true or false. Hence, it can be seen as propositions. The propositional variable P denotes the statement 'all boys are good', Q denotes 'There are at least two students of this class eligible for B.Tech (Honours)' and R denotes 'Some students of this class like all forms of sports'. The propositional variables convey very little information about the actual statement. Phrases like 'all boys', 'at least two', 'some students', etc., are not highlighted in the propositional formula which to some extent quantify the set of elements satisfying some property. Thus, we need to increase the power of propositional logic so that limitations of propositional logic can be addressed to some extent. It is important to highlight that with every logical expression, there is some limitation associated with it, and we are interested in a logical expression that is nearly perfect and close to the given statement.
Predicate logic or First Order Logic (FOL) increases the power of propositional logic or zeroth order logic (ZOL) by using predicates and quantifiers. ZOL is a special case of FOL where there are no quantifiers. Universe of discourse (UOD) or domain is a set under consideration to describe the given argument. Predicates describe the property of elements in UOD and quantifiers describe how many elements in the UOD satisfy the property.
Predicate Logic or First order logic are mathematical assertions containing variables which receive values from a specific domain and become proposition once its variables are assigned values from the respective domain.
The commonly used quantifiers are Universal Quantifier ($\forall$) and Existential Quantifier ($\exists$). The notation $P(x)$ refers to a predicate P such that x in UOD satisfies P. We are interested in knowing how many x in UOD satisfy P. The notation $\forall x P(x)$ says, each element in UOD satisfy P and $\exists x P(x)$ says, some element in UOD satisfy P.
The notation $\forall x$ also mean, 'for each x', 'for every x', 'for any x', 'for arbitrary x' and 'for all x'. Similarly, $\exists x$ also mean, 'for some x', 'at least one x', and 'there exists x'. We do not have notation to refer to expressions such as 'couple of x', 'almost all x', 'many x', 'most of x', 'few of x', and we use $\exists x$ to refer to them.
Let us discuss some examples and their representations in FOL.
The **Universal** quantifier asserts that for all variables in the universe of discourse a given predicate is to be evaluated.
The notation $\forall\ x\ P(x)$ denotes the universal quantification of $P(x)$ and is evaluated to be TRUE if $P(x)$ evaluates to be TRUE for all values of x and FALSE otherwise. As an example,
let $P(x) : x\ \textit{is prime},\ x \in \mathbb{N}$
$Q(x) : x\ \textit{is a non-negative integer},\ x \in \mathbb{N}$
$\forall\ x\ P(x)$ has truthvalue FALSE.
$\forall\ x\ Q(x)$ has truthvalue TRUE.

The **Existential** quantifier ensures that there exists at least one variable x in the universe of discourse such that the predicate can be instantiated on. Also note that if the predicate evaluates to be TRUE for at least one value of x, then existential quantification has a truth value TRUE, and FALSE otherwise. For instance, consider the above predicate, $\exists\ x\ P(x)$ has truth value TRUE.

The **Uniqueness** quantifier denoted as $\exists!$ is read as "there exists exactly one". A uniqueness quantification $\exists!\ x\ P(x)$ is evaluated to TRUE if there exists a unique x for which $P(x)$ is evaluated to TRUE.
For example, $\exists!\ x\ [4x + 3 = 11]$, where $x \in \mathbb{R}$.

Note 1: Let the elements in the domain be $\{x_1, x_2, x_3, \ldots\}$, then,
$\forall\ x\ P(x) \leftrightarrow P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$
Each $P(x_i)$ is a proposition and if each $P(x_i)$ is true then universal quantifier is evaluated to be true.

$\exists\ x\ P(x) \leftrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$

$\exists\ !\ x\ P(x) \leftrightarrow [P(x_1) \wedge \neg P(x_2) \wedge \neg P(x_3) \wedge \neg P(x_4) \wedge \ldots] \vee [P(x_2) \wedge \neg P(x_1) \wedge \neg P(x_3) \wedge \neg P(x_4) \wedge \ldots] \vee [P(x_3) \wedge \neg P(x_1) \wedge \neg P(x_2) \wedge \neg P(x_4) \wedge \ldots] \vee \ldots$

Some Examples:
Consider the universe of discourse as integers. We define the following predicates over integers. We present logical statements below, which are in turn represented using quantifiers.

- $N(x)$: x is a non-negative integer.
- $E(x)$: x is even
- $O(x)$: x is odd
- $P(x)$: x is prime

1. There exist an even integer $\exists\ x\ E(x)$
2. Every integer is even or odd $\forall\ x\ [E(x) \vee O(x)]$
3. All prime integers are non-negative $\forall\ x\ [P(x) \rightarrow N(x)]$
4. There is one and only one even prime $\exists\ !\ x\ [E(x) \wedge P(x)]$
5. The only even prime is two $\forall\ x\ [[E(x) \wedge P(x)] \rightarrow x = 2]$
6. Not all integers are odd $\exists\ x\ \neg O(x)$ or $\neg\forall\ x\ O(x)$
7. Not all primes are odd $\neg\forall\ x\ [P(x) \rightarrow O(x)]$ or $\exists\ x\ [P(x) \wedge \neg O(x)]$

Let us discuss some more examples and their representations in FOL.

1. All boys are good
   - Assuming UOD: set of students and the predicates $BOY(x)$ denotes x is a boy, $x \in UOD$ and $GOOD(x)$ denotes x is good. Then, $\forall x(BOY(x) \rightarrow GOOD(x))$. Note that this expression is more expressive than the simple notation P in propositional logic.
   - Assuming UOD: set of boys, then $\forall x(GOOD(x))$.
   - Assuming UOD: set of things (include living and non-living things). Then, $\forall x(HUMAN(x) \wedge BOY(x) \rightarrow GOOD(x))$, where $HUMAN(x)$ denotes x is human. Thus, for a statement one can get multiple logical expressions.
2. Some boys are good. FOL: $\exists x(BOY(x) \wedge GOOD(x))$. This says, there exists x in UOD such that x is a boy and good.
3. Not all boys are good. FOL: $\neg\forall x(BOY(x) \rightarrow GOOD(x))$. This also means, (i) it is not the case that all boys are good (ii) the statement all boys are good is false (iii) there exists a boy and not good. Therefore, $\exists x(BOY(x) \wedge \neg GOOD(x))$ is equivalent to $\neg\forall x(BOY(x) \rightarrow GOOD(x))$.
4. Some boys are not good. FOL: same as above. $\exists x(BOY(x) \wedge \neg GOOD(x))$.
5. All boys are not good. FOL: $\forall x(BOY(x) \rightarrow \neg GOOD(x))$.
6. None of the boys are good. FOL: $\neg\exists x(BOY(x) \wedge GOOD(x))$. This also means (i) there are no good boys (ii) there exists a good boy is false (iii) all boys are not good

In general, for $x \in UOD$ and a predicate $P(x)$,

- All objects satisfy $P(x)$; denoted as, $\forall x P(x)$
- All objects do not satisfy $P(x)$; denoted as, $\forall x \neg P(x)$
- Not all objects satisfy $P(x)$; denoted as, $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- Some objects satisfy $P(x)$; denoted as, $\exists x P(x)$
- Some objects do not satisfy $P(x)$; denoted as, $\exists x \neg P(x)$
- None of the objects satisfy $P(x)$; denoted as, $\neg \exists x P(x) \equiv \forall x \neg P(x)$

Note:
$\neg \forall\ x\ P(x) \leftrightarrow \exists\ x\ \neg P(x) \qquad \neg \exists\ x\ P(x) \leftrightarrow \forall\ x\ \neg P(x)$

Recall, the notation used to represent 'there exists unique x, $P(x)$' equivalently, 'there exists exactly one x, $P(x)$'.
$\exists ! x P(x)$. Interestingly, this can be expressed using $\forall$ and $\exists$.

$\exists ! x P(x) \equiv \exists x P(x) \wedge \forall y (P(y) \leftrightarrow x = y)$.

The above expression says, there exists x such that for each y if $P(y)$ is true then it implies that $x = y$ and the converse is also true.

Consider the statement "There are at least two students of this class cleared JEE advanced". We shall present below FOL for this expression. Assuming UOD: set of students,

- $\exists x \exists y (x \neq y \wedge CLEARJEEA(x) \wedge CLEARJEEA(y))$, where $CLEARJEEA(x)$ denotes x has cleared JEE Advanced. When we parse this expression from left to right we observe the following; (i) $\exists x$ says there exists at least one x satisfying some property (ii) $\exists y$ also says at least one y and at this point, x and y can be same. (iii) the expression $x \neq y$ says they are different and hence at least two students exist satifying the predicate.
- Since UOD is a set and the students who cleared JEE advanced is a subset of UOD, the expression |cleared JEE advanced | ≥ 2 is true. However, this expression is not a logical expression, and cannot be considered as FOL.
- Similarly, |UOD|−|not cleared JEE advanced| ≥ 2, not a logical expression as it uses relational operators. It is important to note that FOL works with elements of UOD and their properties and not on UOD itself, as a set or its subsets.

Consider the statement "There exists a boy who is taller than every other student". From the given statement, one can infer that there will be exactly one boy who is taller than every other student. This means that we cannot have two students of same height. Assuming UOD: set of students, we shall present below FOL for this expression.

- $\exists x (BOY(x) \wedge \forall y (y \neq x \rightarrow TALL(x, y))$. $TALL(x, y)$ denotes x is taller than y, if heights of x and y are same or height of x is smaller than height of y, then $TALL(x, y)$ returns false. This is a valid logical expression as the statement is true if there exists exactly one x who is the tallest among students.
- $\exists x (BOY(x) \wedge \forall y (y \neq x \wedge TALL(x, y))$. This says, x is a boy and for each y in UOD, if y is different from x, then x is taller than y. Note the usage of '$\wedge$' in '$\forall$' and in the above expression, we have used '$\rightarrow$' in '$\forall$'. Since UOD is the set of students, this expression is also valid.

- $\exists ! x HEIGHT(x) > \forall y HEIGHT(y)$ - Not a valid logical expression as it uses relational operator.
- $\exists x(T(x,x_1) \wedge T(x,x_2) \wedge T(x,x_3) \wedge, \ldots, T(x,x_n))$, $T(x,x_i)$ denotes x is taller than x_i. This is a valid expression. If there are two students with same height, then this expression returns false, otherwise true.
- $\exists ! x(T(x) \wedge \forall y(y \neq x \rightarrow \neg T(y))$. - This is not valid as x is a boy is not highlighted. Note that $T(x)$ denotes x is the tallest in class. $\exists ! x(T(x) \wedge BOY(x) \wedge \forall y(y \neq x \rightarrow \neg T(y))$ is a valid expression.
- $\exists ! x(TALLEST(x))$ - This is not valid as x is a boy is not highlighted. Modified expression: $\exists ! x(BOY(x) \wedge TALLEST(x))$ is valid.

`Remarks:`
1. Note that for the statement "all boys are good", the valid expression is $\forall x(BOY(x) \rightarrow GOOD(x))$. Let us analyze the intuition behind using implication operator for 'for all' quantifier.
2. The expression $\forall x(BOY(x) \wedge GOOD(x))$ says that each element in UOD is a boy and good, equivalently, all are good boys. Clearly, this expression does not express "all boys are good".
3. When we use $\forall x(BOY(x) \rightarrow GOOD(x))$ to express "all boys are good", Does the expression evaluates to true even when x is not a boy. Answer: yes.

`Justification for the use of implication operator in` $\forall$ `and` $\wedge$ `operator in` $\exists$

1. For the statement "all boys are good", usage of operator $\wedge$ or $\vee$ is incorrect and expresses a different meaning than the context.
2. In the context of $\forall x$, when we say $\forall x(P(x) \rightarrow Q(x))$ is true, it means two things; (i) both $P(x)$ and $Q(x)$ are true (ii) $\neg P(x)$ is true and $Q(x)$ is true or false. That is, $(P \rightarrow Q) \rightarrow (\neg P \rightarrow Q) \vee (\neg P \rightarrow \neg Q)$. Note that this expression is a tautology. Therefore, when $P \rightarrow Q$ is used in the context, it implicitly means that $(\neg P \rightarrow Q) \vee (\neg P \rightarrow \neg Q)$ is true. Let us revisit our example "all boys are good". This expression also means that all girls are good or not good, which is always true. We also highlight that there is no logical operator which expresses just "all boys are good" and does not express any other hidden meaning. Since implication operator is 'near perfect' operator, we always use implication according to the context. Further, this operator does not convey any other meaning which is logically incorrect.
3. For the statement "some boys are good", the apt operator is $\wedge$ and the logical expression is $\exists x(BOY(x) \wedge GOOD(x))$. Suppose, we use implication operator in place of 'logical AND' operator. Then, the expression is $\exists x(BOY(x) \rightarrow GOOD(x))$. This implicitly means that some girls are good or some girls are not good which cannot be inferred from some boys are good. For example; the set of girls may be empty. Therefore, $\rightarrow$ operator is not the perfect operator for $\exists$ quantifier.
4. We highlight that if the context under discussion demands the usage of $\rightarrow$ in $\exists$ and $\wedge$ in $\forall$, then it must be used. For example; we express the statement "there exists a boy such that if he is eligible for distinction, then he is eligible for honours" as $\exists x(BOY(x) \wedge DIST(x) \rightarrow HONO(x))$.
5. The statement "Every natural number is a real number and a rational number" can be expressed as $\forall x(REAL(x) \wedge RAT(x))$.

Note 2: The scope of a quantifier is that part of an assertion in which variables are bounded by the quantifier.
$\forall\, x\, [P(x) \vee Q(x)] \not\leftrightarrow \forall\, x\, P(x) \vee Q(x) \not\leftrightarrow P(x) \vee \forall\, x\, Q(x)$

$\forall\ x\ [P(x) \vee Q(y)] \leftrightarrow \forall\ x\ P(x) \vee \forall\ x\ Q(y)$
$\exists\ x\ [P(x) \wedge Q] \leftrightarrow \exists\ x\ P(x) \wedge Q$

Compound statements involving predicates
For every pair of integers x and y there exists a z such that $x + z = y$
The above predicate is represented as $\forall\ x\ \forall\ y\ \exists\ z\ [x + z = y]$
Note that if universe of discourse is integer, then the predicate's truth value is TRUE.
If universe of discourse is $\mathbb{N}$, then the predicate is FALSE for some predicate constants.
i.e., $\neg\forall\ x\ \forall\ y\ \exists\ z\ [x + z = y] \implies \exists\ x\ \exists\ y\ \forall\ z\ \neg[x + z = y] \implies \exists\ x\ \exists\ y\ \forall\ z\ [x + z \neq y]$, in particular, $x = 4, y = 1$ and for any z, $x + z \neq y$.

Also observe from the previous example, the negation operator flips the quantifier from universal to existential and vice versa.

Nested Quantifiers:
We shall now discuss the expressions involving multiple quantifiers. In this section, we consider the following case studies and express them using predicate logic.

`Definition of limit:`

$$\underset{x \to c}{Lt}\ \text{f(x)=k} \leftrightarrow \forall\epsilon_{\epsilon>0}\ \exists\delta_{\delta>0}\ \forall x\ [(|x - c| < \delta) \to (|f(x) - k| < \epsilon)].$$

`Negation of the above limit is defined as ,`

$$\underset{x \to c}{Lt}\ \text{f(x)}\neq\text{k} \leftrightarrow \exists\epsilon_{\epsilon>0}\ \forall\delta_{\delta>0}\ \exists x\ [(|x - c| < \delta) \wedge (|f(x) - k| \geq \epsilon)].$$

Logical representation of statements using nested quantifiers.

– `The sum of two positive integers is always positive.`

$\forall x\ \forall y\ (x > 0 \wedge y > 0 \to x + y > 0)$

– `For every real number except` 0`, there exists a multiplicative inverse.`

$\forall x\ (x \neq 0 \to \exists y\ (xy = 1))$

– `If a person is female and is a parent, then this person is someone's mother.`

$\forall x\ [female(x) \wedge parent(x) \to \exists y\ (mother(x, y))]$

– `Every train is faster than some cars.`

$\forall x\ [train(x) \to \exists y\ (car(y) \wedge faster(x, y))]$

– `Some cars are slower than all trains but at least one train is faster than every car.`

$\exists x\ [car(x) \wedge \forall y\ (train(y) \to slower(x, y))] \wedge \exists x\ [train(x) \wedge \forall y\ (car(y) \to faster(x, y))]$

– `If it rains tomorrow, then somebody will get wet.`

$P \rightarrow \exists x\ (person(x) \wedge wet(x))$

Let A be a 2 dimensional integer array with 20 rows (indexed from 1 to 20) and 30 columns (indexed 1 to 30). Using first order logic make the following assertions.

1. `All entries of` A `are non-negative.`
   $\forall i\ \forall j\ (1 \leq i \leq 20, 1 \leq j \leq 30 \rightarrow A[i][j] \geq 0)$

2. `All entries of` 4^{th} `and` 15^{th} `rows are positive.`
   $\forall j\ (1 \leq j \leq 30 \rightarrow (A[4][j] \geq 1) \wedge (A[15][j] \geq 1))$

3. `Some entries of` A `are zero.`
   $\exists i\ \exists j\ (1 \leq i \leq 20, 1 \leq j \leq 30 \wedge A[i][j] = 0)$

4. `Entries of` A `are sorted in row major order. (i.e., the entries are in order within rows and every entry of the` i^{th} `row is less than or equal to every entry of the` $(i+1)^{th}$ `row).`
   $\{\forall i\ \forall j\ [1 \leq i \leq 20, 1 \leq j \leq 29 \rightarrow A[i][j] \leq A[i][j+1]] \wedge [\forall i\ 1 \leq i \leq 19 \rightarrow A[i][30] \leq A[i+1][1]]\}$

Logical Identities

1. $\forall\ x\ P(x) \rightarrow P(c)$, for some constant c.
2. For some c, $P(c) \rightarrow \exists\ x\ P(x)$
3. $\forall\ x\ \neg P(x) \leftrightarrow \neg\exists\ x\ P(x)$
4. $\forall\ x\ P(x) \rightarrow \exists\ x\ P(x)$
5. $\exists\ x\ \neg P(x) \leftrightarrow \neg\forall\ x\ P(x)$
6. $\forall\ x\ P(x) \wedge Q \leftrightarrow \forall\ x\ [P(x) \wedge Q]$
7. $\forall\ x\ P(x) \wedge \forall\ x\ Q(x) \leftrightarrow \forall\ x\ [P(x) \wedge Q(x)]$
8. $\forall\ x\ P(x) \vee \forall\ x\ Q(x) \rightarrow \forall\ x\ [P(x) \vee Q(x)]$
9. $\exists\ x\ [P(x) \wedge Q(x)] \rightarrow \exists\ x\ P(x) \wedge \exists\ x\ Q(x)$
10. $\exists x\ P(x) \vee \exists\ x\ Q(x) \leftrightarrow \exists\ x\ [P(x) \vee Q(x)]$

Rules of Inference for Quantified Statements

Rule of Inference	Name
$\forall x\ P(x) \implies P(c)$	Universal Instantiation
$P(c)$ for any arbitrary c $\implies \forall x\ P(x)$	Universal Generalization
$\exists x\ P(x) \implies P(c)$ for some c	Existential Instantiation
$P(c)$ for some c $\implies \exists x\ P(x)$	Existential Generalization

Claim. $\forall\ x\ P(x) \wedge \forall\ x\ Q(x) \leftrightarrow \forall\ x\ [P(x) \wedge Q(x)]$

Proof. Necessity: It follows from definition that $[P(x_0) \wedge P(x_1) \wedge P(x_2) \wedge \ldots] \wedge [Q(x_0) \wedge Q(x_1) \wedge Q(x_2) \wedge \ldots]$ where $\{x_0, x_1, x_2, \ldots\}$ is the universe of discourse.
Apply the below rules inductively. For simplicity we work with the first two terms.
$[P(x_0) \wedge P(x_1)] \wedge [Q(x_0) \wedge Q(x_1)]$

Due to Associativity, $\implies [P(x_0) \wedge P(x_1) \wedge Q(x_0)] \wedge Q(x_1)$
Due to Commutativity, $\implies [P(x_0) \wedge Q(x_0) \wedge P(x_1)] \wedge Q(x_1)$
Due to Associativity, $\implies [P(x_0) \wedge Q(x_0)] \wedge [P(x_1) \wedge Q(x_1)]$
Once inductive application of the above rules is completed in order for all elements in the universe of discourse, we get
$(P(x_0) \wedge Q(x_0)) \wedge (P(x_1) \wedge Q(x_1)) \wedge \ldots$
By definition, $\implies \forall x\ [P(x) \wedge Q(x)]$. Necessity follows.
Sufficiency: $\forall x\ [P(x) \wedge Q(x)]$
By definition, $\implies (P(x_0) \wedge Q(x_0)) \wedge (P(x_1) \wedge Q(x_1)) \wedge \ldots$
Apply the below rules inductively. For simplicity we work with the first two terms.
Consider $[P(x_0) \wedge Q(x_0)] \wedge [P(x_1) \wedge Q(x_1)]$
Due to Associativity, $\implies [P(x_0) \wedge Q(x_0) \wedge P(x_1)] \wedge Q(x_1)$
Due to Commutativity, $\implies [P(x_0) \wedge P(x_1) \wedge Q(x_0)] \wedge Q(x_1)$
Due to Associativity, $\implies [P(x_0) \wedge P(x_1)] \wedge [Q(x_0) \wedge Q(x_1)]$
It follows that $[P(x_0) \wedge P(x_1) \wedge P(x_2) \wedge \ldots] \wedge [Q(x_0) \wedge Q(x_1) \wedge Q(x_2) \wedge \ldots]$
From necessity and sufficiency, the claim follows. □

Claim. $\exists\ x\ [P(x) \vee Q(x)] \leftrightarrow \exists x\ P(x) \vee \exists\ x\ Q(x)$

Proof. From previous claim $\forall\ x\ [P(x) \wedge Q(x)] \leftrightarrow \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$
Inverse, $\implies \neg\forall\ x\ [P(x) \wedge Q(x)] \leftrightarrow \neg[\forall\ x\ P(x) \wedge \forall\ x\ Q(x)]$
De-Morgans law, $\implies \exists x\ [\neg[P(x) \wedge Q(x)]] \leftrightarrow \neg\forall\ x\ P(x) \vee \neg\forall\ x\ Q(x)$
De-Morgans law, $\implies \exists x\ [\neg P(x) \vee \neg Q(x)] \leftrightarrow \exists\ x\ \neg P(x) \vee \exists\ x\ \neg Q(x)$
$R(x) = \neg P(x)$ and $S(x) = \neg Q(x) \implies \exists x\ [R(x) \vee S(x)] \leftrightarrow \exists\ x\ R(x) \vee \exists\ x\ S(x)$ □

A second proof for Claim 1

Claim. $\forall\ x\ [P(x) \wedge Q(x)] \leftrightarrow \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$

Proof. Necessity: $\forall x \in UOD$, $P(x) \wedge Q(x)$ is TRUE.
$\implies P(x)$ is TRUE and $Q(x)$ is TRUE
$\implies \forall\ x\ P(x) \wedge \forall\ x\ Q(x)$.
Sufficiency: $\forall\ x\ P(x) \wedge \forall\ x\ Q(x)$ is TRUE.
On Simplification, $\implies \forall\ x\ P(x)$ is TRUE.
On Simplification, $\implies \forall\ x\ Q(x)$ is TRUE.
For any $x \in UOD$, $P(x) \wedge Q(x)$ is TRUE.
$\implies \forall x\ [P(x) \wedge Q(x)]$. □

Claim. $\exists\ x\ [R(x) \vee S(x)] \leftrightarrow \exists x\ R(x) \vee \exists\ x\ S(x)$

Proof. $\exists\ x\ [R(x) \vee S(x)]$, let $x = c \in UOD$
Existential instantiation, $\implies R(c) \vee S(c)$ is TRUE.
Existential generalization, $\implies \exists x\ R(x) \vee \exists\ x\ S(x)$
Sufficiency: $\exists x\ R(x) \vee \exists\ x\ S(x)$ let $x = c \in UOD$
Existential instantiation, $\implies R(c) \vee S(c)$ is TRUE. Note that $\exists\ x\ S(x)$ may be true for $x = d$ and may not be true for $x = c$. However, it is certainly true for $\exists\ x\ R(x)$. Because of logical 'or', the truth value is 'True' for the expression even if $S(d)$ is 'false'.
Existential generalization, $\implies \exists x\ [R(x) \vee S(x)]$ □

Claim. $\forall\ x\ P(x) \vee \forall\ x\ Q(x) \rightarrow \forall\ x\ [P(x) \vee Q(x)]$

Proof. $\forall\ x\ P(x) \vee \forall\ x\ Q(x)$, let $x = c \in UOD$
Universal instantiation, $\implies P(c) \vee Q(c)$
Case 1: $P(c)$ is TRUE and $Q(c)$ is TRUE
$\implies P(c) \vee Q(c)$ is TRUE
Case 2: $P(c)$ is TRUE and $Q(c)$ is FALSE
$\implies P(c) \vee Q(c)$ is TRUE
Case 3: $P(c)$ is FALSE and $Q(c)$ is TRUE
$\implies P(c) \vee Q(c)$ is TRUE
Case 1,2,3, $\implies P(c) \vee Q(c)$ is TRUE
Universal generalization, $\implies \forall\ x\ [P(x) \vee Q(x)]$ □

Converse of the above claim does not hold. i.e., $\forall\ x\ [P(x) \vee Q(x)] \not\rightarrow \forall\ x\ P(x) \vee \forall\ x\ Q(x)$
Counter example:
$P(x) : x$ is an irrational number.
$Q(x) : x$ is a rational number.
UOD: $\mathbb{R}$
Questions:

- Which is correct $\exists y\ \forall x\ [x + y = 0]$ or $\forall x\ \exists y\ [x + y = 0]$ where $x, y \in \mathbb{R}$ **?**

Claim. $\exists x(P(x) \rightarrow Q(x)) \leftrightarrow \exists x\ P(x) \rightarrow \exists x\ Q(x)$. Is the claim TRUE ?

$\exists x(P(x) \rightarrow Q(x)) \leftrightarrow \exists x(\neg P(x) \vee Q(x))$
$\leftrightarrow \exists x(\neg P(x)) \vee \exists x(Q(x))$
$\leftrightarrow \neg\forall x\ P(x) \vee \exists x\ Q(x)$
$\leftrightarrow \forall x\ P(x) \rightarrow \exists x\ Q(x)$

Answering the above question is equivalent to checking the necessary and sufficiency conditions of

$\forall x\ P(x) \rightarrow \exists x\ Q(x) \leftrightarrow \exists x\ P(x) \rightarrow \exists x\ Q(x)$

We shall now verify the validity of the above statement using a truth table. Note that like propositional variables, we treat predicate variables as a variable assuming the value of the variable is either true or false.

Truth Table

$\forall x\ P(x)$	$\exists x\ P(x)$	$\exists x\ Q(x)$	$\forall x\ P(x) \rightarrow \exists x\ Q(x)$	$\exists x\ P(x) \rightarrow \exists x\ Q(x)$
0	0	0	1	1
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	N.A.	N.A.
1	0	1	N.A.	N.A.
1	1	0	0	0
1	1	1	1	1

Also note that in fifth row, it is not possible to have 'true' for $\forall x$ and 'false' for $\exists x$ and this impossibility is written as N.A (not applicable). From the last two columns, it is clear that sufficiency part is TRUE, and necessity is FALSE.

i.e., $[\exists x\ P(x) \to \exists x\ Q(x)] \to [\forall x\ P(x) \to \exists x\ Q(x)]$

$[\forall x\ P(x) \to \exists x\ Q(x)] \not\to [\exists x\ P(x) \to \exists x\ Q(x)]$

For disproving the necessity, consider the following counter example.
P(x): $x = 2$.
Q(x): $x \neq x$.
UOD: integers.
Note that $\forall x\ P(x)$ is FALSE, $\exists x\ P(x)$ is TRUE, and $\exists x\ Q(x)$ is FALSE. It is clear that the premise is true and the conclusion is false, therefore the necessity is false.

Logical Inference from a given Argument

In this section, we shall determine the truth value of a logical argument using logical identities and logical implications. We first transform the argument into logical notation, followed by validity checking using laws presented in the previous section.

1. All philosophers are scientists.
   All scientists are engineers.
   Therefore, all philosophers are engineers.
   In FOL, the above arguemnt is translated into
   $[\forall x(P(x) \to S(x)) \wedge \forall x(S(x) \to E(x))] \to \forall x(P(x) \to E(x))$
   U.I of the premise gives;
   $P(a) \to S(a)$, for any a
   $S(a) \to E(a)$, for any a
   We know that $(P \to Q) \wedge (Q \to R) \to (P \to R)$.
   Thus, we get, $P(a) \to E(a)$. Since a is arbitrary, on UG,
   $\forall x(P(x) \to E(x))$, the desired claim.

2. All philosophers are scientists.
   some scientists are engineers.
   Therefore, some philosophers are engineers.
   In FOL, the above arguemnt is translated into
   $[\forall x(P(x) \to S(x)) \wedge \exists x(S(x) \wedge E(x))] \to \exists x(P(x) \wedge E(x))$
   The claim is false. Consider the Venn diagram with three sets A : philosophers B : scientists C : engineers
   Consider the scenario in which $A \subset B$ and C is having intersection with B but not with A. For this illustration, the premise is true and the conclusion is false.

3. All philosophers are scientists.
   Some are not scientists.
   Therefore, some are not philosophers.
   In FOL, the above arguemnt is translated into
   $[\forall x(P(x) \rightarrow S(x)) \wedge \exists x(\neg S(x))] \rightarrow \exists x(\neg P(x))$
   U.I of the premise gives;
   $P(a) \rightarrow S(a)$, for any a
   E.I of the premise gives;
   $\neg S(a)$, for some a
   The contrpositive of $P(a) \rightarrow S(a)$ gives $\neg S(a) \rightarrow \neg P(a)$.
   Further, $\neg S(a) \wedge (\neg S(a) \rightarrow \neg P(a))$ gives $\neg P(a)$, thus the claim follows.

4. All philosophers are scientists.
   some are not scientists.
   some are engineers.
   Therefore, some philosophers are engineers.
   In FOL, the above arguemnt is translated into
   $[\forall x(P(x) \rightarrow S(x)) \wedge \exists x(\neg S(x)) \wedge \exists x(E(x))] \rightarrow \exists x(P(x) \wedge E(x))$
   The claim is false. Consider the Venn diagram with three sets A : philosophers B : scientists C : engineers
   Consider the scenario in which $A \subset B$ and there is an element of UOD outside B denoting some are not scientists. $C \subset B$ not having intersection with A. For this illustration, the premise is true and the conclusion is false.

Question 1. If a teacher teaches DM or DSA, then he is considered to be a TCS teacher. If he is a TCS teacher, then he teaches GT. He does not teach GT. Therefore, he does not teach DSA.
A: teaches DM
B: teaches DSA
C: TCS teacher
D: teaches GT

$$
\begin{array}{ll}
(A \vee B) \rightarrow C & \ldots (1) \\
C \rightarrow D & \ldots (2) \\
\neg D & \ldots (3) \\
\therefore \neg B &
\end{array}
$$

$Proof$

$$
\begin{array}{lll}
\text{From} 1, 2: & (A \vee B) \rightarrow D & \ldots (4) - \text{Due to Hypothetical Syllogism.} \\
3, 4: & \neg(A \vee B) & \ldots (5) \\
5: & \neg A \wedge \neg B & \ldots (6) \\
6: & \neg B & QED
\end{array}
$$

Therefore, the conclusion, teacher does not teach DSA follows from the given logical argument.

Question 2. Derive a contradiction for the premises 1-5.

$$
\begin{array}{ll}
A \to B \lor C & \ldots (1) \\
D \to \neg C & \ldots (2) \\
B \to \neg A & \ldots (3) \\
A & \ldots (4) \\
D & \ldots (5)
\end{array}
$$

$$
\begin{array}{lll}
1,4: & B \lor C & \ldots (6) \\
2,5: & \neg C & \ldots (7) \\
6: & \neg C \to B & \ldots (8) \\
7,8: & B & \ldots (9) \\
3: & A \to \neg B & \ldots (10) \\
4,10: & \neg B & \ldots (11) \\
9,11: & B \land \neg B & \textit{a contradiction}
\end{array}
$$

Therefore, the given argument is logically inconsistent.

Question 3. Show that $R \lor S$ follows logically from the premises

$$
\begin{array}{ll}
C \lor D & \ldots (1) \\
C \lor D \to \neg H & \ldots (2) \\
\neg H \to A \land \neg B & \ldots (3) \\
A \land \neg B \to R \lor S & \ldots (4)
\end{array}
$$

$$
\begin{array}{lll}
1,2: & \neg H & \ldots (5) \\
3,5: & A \land \neg B & \ldots (6) \\
4,6: & R \lor S & QED
\end{array}
$$

Question 4. Show that $S \lor R$ follows logically from the first three premises.

$$
\begin{array}{ll}
P \lor Q & \ldots (1) \\
P \to R & \ldots (2) \\
Q \to S & \ldots (3)
\end{array}
$$

$$
\begin{array}{lll}
1: & \neg Q \to P & \ldots (4) \\
2,4: & \neg Q \to R & \ldots (5) \\
5: & \neg R \to Q & \ldots (6) \\
3,6: & \neg R \to S & \ldots (7) \\
7 & S \lor R & QED
\end{array}
$$

Question 5. If Jack misses many classes through illness, then he fails high school.
If Jack fails high school, then he is uneducated.
If Jack reads a lot of books, then he is not uneducated.
Jack misses many classes through illness and reads a lot of books.

Check whether the argument is consistent.

J: Jack misses many classes through illness.
H: Jack fails high school.
U: Jack is uneducated.
R: Jack reads a lot of books.

$$\begin{array}{ll}
 & J \to H \quad \ldots (1) \\
 & H \to U \quad \ldots (2) \\
 & R \to \neg U \ldots (3) \\
 & J \wedge R \quad\;\; \ldots (4) \\
Proof: & \\
1,2: & J \to U \quad \ldots (5) \\
3: & U \to \neg R \ldots (6) \\
5,6: & J \to \neg R \ldots (7) \\
7: & R \to \neg J \ldots (8) \\
4: & R \qquad\quad \ldots (9) \\
8,9: & \neg J \qquad\;\; \ldots (10) \\
4: & J \qquad\quad\; \ldots (11) \\
10,11: & J \wedge \neg J \qquad \textit{a contradiction}
\end{array}$$

Therefore, the given argument is logically inconsistent.

Assertions Involving Quantifiers - Validity Checking

Question 1. *Check validity*

A student in this class has not read the book and everyone in this class passed the first exam. Therefore someone who passed the first exam has not read the book.

Representation using logic variables
$C(x)$: x is in this class.
$P(x)$: x passes the first exam.
$B(x)$: x has read the book.

We want to prove $\exists x\ (P(x) \wedge \neg B(x))$ from 1 and 2.

$$\begin{array}{lll}
 & \exists x\ (C(x) \wedge \neg B(x)) & \ldots (1) \\
 & \forall x\ (C(x) \to P(x)) & \ldots (2) \\
EI\ of\ 1 & C(a) \wedge \neg B(a) & \ldots (3) \\
3 & C(a) & \ldots (4) \\
EI\ of\ 2 & C(a) \to P(a) & \ldots (5) \\
4,5: & P(a) & \ldots (6) \\
3 & \neg B(a) & \ldots (7) \\
6,7: & P(a) \wedge \neg B(a) & \ldots (8) \\
EG\ of\ 8: & \exists x\ (P(x) \wedge \neg B(x)) & QED
\end{array}$$
[2]

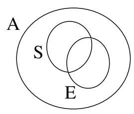
Question 2. *Check validity*

Some scientists are not engineers.
Some astronauts are not engineers.

[2] EI: Existential Instantiation
UI: Universal Instantiation
EG: Existential Generalization
UG: Universal Generalization 3

Hence, some scientists are not astronauts.

Representation using logic variables
$E(x)$: x is an engineer.
$S(x)$: x is a scientist.
$A(x)$: x is an astronaut.
To prove or disprove $\exists x\ (S(x) \wedge \neg A(x))$ from 1 and 2.



A counter example.

$\exists x\ (S(x) \wedge \neg E(x)) \ldots (1)$
$\exists x\ (A(x) \wedge \neg E(x)) \ldots (2)$

Counter example is shown using the Venn diagram. Note that, if we want to prove a claim using Venn diagram then we have to enumerate all possible Venn diagrams in the given context and therefore, such a proof method is not advisable as our listing may not be exhaustive. However, to disprove a claim, the existence of even one Venn diagram suffices. Due to this reasoning, we use Venn diagram to disprove a claim and a proof using logical identities if the claim is correct.

Question 3. *Check validity*

All integers are rational numbers.
Some integers are powers of 2.
Therefore, some rational numbers are powers of 2.
Representation using logic variables
$I(x)$: x is an Integer.
$R(x)$: x is a rational number.
$P(x)$: x is a power of 2.
To prove or disprove $\exists x\ (R(x) \wedge P(x))$ from 1 and 2.

$$
\begin{array}{lll}
 & \forall x\ (I(x) \rightarrow R(x)) & \ldots (1) \\
 & \exists x\ (I(x) \wedge P(x)) & \ldots (2) \\
UI\ of\ 1 & I(a) \rightarrow R(a) & \ldots (3) \\
EI\ of\ 2 & I(a) \wedge P(a) & \ldots (4) \\
4 & I(a) & \ldots (5) \\
3,5: & R(a) & \ldots (6) \\
4 & P(a) & \ldots (7) \\
6,7: & R(a) \wedge P(a) & \ldots (8) \\
EG\ of\ 8: & \exists x\ (R(x) \wedge P(x)) & QED
\end{array}
$$

Question 4. *Check validity*

Premise: $\exists x\ (F(x) \wedge S(x)) \rightarrow \forall y\ (H(y) \rightarrow W(y))$
$\exists y\ (H(y) \wedge \neg W(y))$
Conclusion: $\forall x\ (F(x) \rightarrow \neg S(x))$

	$\exists x\ (F(x) \wedge S(x)) \rightarrow \forall y\ (H(y) \rightarrow W(y))$	$\ldots (1)$
	$\exists y\ (H(y) \wedge \neg W(y))$	$\ldots (2)$
2	$\neg\neg\exists y\ (H(y) \wedge \neg W(y))$	$\ldots (3)$
3	$\neg\forall y\ \neg(H(y) \wedge \neg W(y))$	$\ldots (4)$
4	$\neg\forall y\ (H(y) \rightarrow W(y))$	$\ldots (5)$
$1, 5 :$	$\neg\exists x\ (F(x) \wedge S(x))$	$\ldots (6)$
6	$\forall x\ (F(x) \rightarrow \neg S(x))$	QED

Question 5. Show that $\forall x\ (P(x) \vee Q(x)) \rightarrow \forall x\ P(x) \vee \exists x\ Q(x)$

Proof by contradiction: Assume on the contrary that the conclusion is FALSE. i.e., include $\neg$ *Conclusion* as part of premise.

premise	$\forall x\ (P(x) \vee Q(x))$	$\ldots (1)$
premise assumed	$\neg[\forall x\ P(x) \vee \exists x\ Q(x)]$	$\ldots (2)$
2	$\neg\forall x\ P(x) \wedge \neg\exists x\ Q(x)$	$\ldots (3)$
3	$\exists x\ \neg P(x) \wedge \forall x\ \neg Q(x)$	$\ldots (4)$
4	$\exists x\ \neg P(x)$	$\ldots (5)$
EI of 5	$\neg P(a)$	$\ldots (6)$
4	$\forall x\ \neg Q(x)$	$\ldots (7)$
UI of 7	$\neg Q(a)$	$\ldots (8)$
$7, 8$	$\neg P(a) \wedge \neg Q(a)$	$\ldots (9)$
9	$\neg[P(a) \vee Q(a)]$	$\ldots (10)$
UI of 1	$P(a) \vee Q(a)$	$\ldots (11)$
$10, 11$	$\neg[P(a) \vee Q(a)] \wedge [P(a) \vee Q(a)]$	*a contradiction*

Therefore our assumption is wrong/FALSE and conclusion is TRUE. Therefore $\forall x\ P(x) \vee \exists x\ Q(x)$ follows from $\forall x\ (P(x) \vee Q(x))$. $\square$

Having learnt First Order Logic (FOL), the following Qualities are expected out of the learner by the end of First Order Logic learning.

1. *Be precise and concise.*
2. *Always think before speak.*
3. *Be consistent - always think before you speak.*
4. *Stop believing - start asking/looking for logical reasoning/proof*
5. *Start using necessary, sufficiency, if and only if, during conversations.*

Food for thought
Express the following using First order Logic.

1. *Among the institute faculty, there exists a set of faculty whose expertise is computer science.*
2. *Among the school kids, there exists a set of kids whose IQ level number is five.*

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