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COM205T Discrete Structures for Computing-Lecture Notes

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Relations

Objective: In this module, we shall introduce sets, their properties, and relationships among their elements. We shall also look at in detail the properties of relations. Further we also count sets that satisfy specific properties.

Basic Definitions

A set (universe of discourse) is a *well defined* collection of distinct objects.

Cross Product

Let A and B be two sets. The cross product $A \times B$ is a set, defined as $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

In general, given the sets $A_1, A_2, \ldots, A_n$, we can define $A_1 \times A_2 \times \ldots \times A_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i, 1 \leq i \leq n\}$.

Note that if $|A| = n$, then $|A \times A| = n^2$, and $|A_1 \times A_2 \times \ldots \times A_n| = m_1 \cdot m_2 \cdots m_n$ where $|A_i| = m_i$, $1 \leq i \leq n$.

Note: Empty set ϕ is not well defined. If $B = \phi$, then $A \times B = \phi$, is also not well defined. Hence, the only relation possible is empty relation.

Relation

Let A and B be two sets. A binary relation R from A to B is a set, defined as $R \subseteq A \times B$. For a set A and the cross product $A \times A$, a binary relation R defined on A is such that $R \subseteq A \times A$. A ternary relation R on A is such that $R \subseteq A \times A \times A$. In general, an n-ary relation R on A is such that $R \subseteq A \times A \times \ldots \times A$ (n times). Interestingly, unary relation exists and it is a subset of A. That is, an unary relation R on A is such that $R \subseteq A$. Note that any binary relation is a subset of $A \times A$ and a ternary relation is a subset of $A \times A \times A$.

Example 1.
Let $A = \{1, 2, 3\}$. R_0 is an example unary relation. R_1 to R_4 are binary relations defined on A.
$R_0 = \{2, 3\}$
$R_1 = \{(1,1), (2,2), (3,3)\}$ $\quad$ $R_2 = \{(1,1), (2,1), (3,2)\}$
$R_3 = \phi$ $\quad$ $R_4 = A \times A$

Example 2.
$S = \{$DM,DSA,ALG,OOP,C++,JAVA$\}$.
$R_5 = \{(x, y) \mid x, y \in S$ and x is a prerequisite for $y\}$.
$R_5 = \{$(DSA,ALG), (DM,ALG), (DM,OOP), (OOP,C++)$\}$.

Example 3. Let $I = \{i_1, i_2, \ldots, i_m\}$ be the items in a supermarket. A ternary relation R_3 on I is defined as $R_3 = \{(i_1, i_4, i_3), (i_2, i_1, i_7), (i_7, i_8, i_9)\}, R_3 \subseteq I \times I \times I$.

Remark: Familiar examples for relations are tables in databases or a spreadsheet (excel sheet). A row (or the entire spreadsheet) corresponds to an element in the underlying cross product of items (columns) and each row is a relation R by defintion and each row is an element in R. Further, each row is a transaction.

Claim Consider a set A with $|A| = n$. The size of the powerset (the set of all subsets) is 2^n.

Proof 1: By definition, the set of all subsets of A include subsets of size '0', '1', '2', and so on. The number of subsets of size i is $\binom{n}{i}$. Therefore, the total number of subsets is $\sum_{i=0}^{i=n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}$, which is precisely 2^n.

Proof 2: Consider a subset B, observe that each element of A is either present or not in B. Thus, to get all subsets, there are two possibilities (present or not) for each element in A. Therefore, 2^n subsets.

Proof 3: Note that there are 2^n possible binary strings of length n. Further, the binary string '1101' corresponds to the set $\{4, 3, 1\}$, i.e., the presence of '1' indicates the presence of the corresponding element in the subset and '0' otherwise. This shows that there is one-one mapping between the set of binary strings and the set of all subsets. Thus, there are 2^n subsets.

Proof 4: Proof by mathematical induction on n. $n = 0$. For an empty set, the powerset is also an empty set and the cardinality of powerset is one. Therefore, $2^0 = 1$ is true. Assume a set of size n has 2^n subsets. Consider a set $\{a_1, \ldots, a_n, a_{n+1}\}$ of size $(n + 1)$. By induction hypothesis the set $\{a_1, \ldots, a_n\}$ has 2^n subsets and all of which will remain subsets for the set $\{a_1, \ldots, a_n, a_{n+1}\}$. In addition, we obtain new subsets by augmenting the element a_{n+1} to each subset of $\{a_1, \ldots, a_n\}$. Thus, there are 2^n subsets not containing a_{n+1} and 2^n subsets containing a_{n+1}. Overall, 2^{n+1} subsets and therefore, the claim follows.

Claim. Consider a set A with $|A| = n$. The maximum number of binary relations on A is 2^{n^2}.

Proof. Let $A \times A = \{x_1, x_2, \ldots, x_{n^2}\}$. A relation is a subset of $A \times A$. The number of subsets of a set of size n^2 is 2^{n^2}. Thus, the claim follows. □

A **Unary** relation R on A is $R \subseteq A$. The maximum number of unary relations is 2^n.

Properties of Relations

Consider a binary relation $R \subseteq A \times A$, we define the following on R
(1) R is *reflexive* iff $\forall a \in A, ((a, a) \in R)$
(2) R is *symmetric* iff $\forall a, b \in A, ((a, b) \in R \rightarrow (b, a) \in R)$
(3) R is *transitive* iff $\forall a, b, c \in A, ((a, b) \in R$ and $(b, c) \in R \rightarrow (a, c) \in R)$
(4) R is *asymmetric* iff $\forall a, b \in A, ((a, b) \in R \rightarrow (b, a) \notin R)$

(5) R is *antisymmetric* iff $\forall a, b \in A, [((a, b) \in R \wedge (b, a) \in R) \rightarrow a = b]$
(6) R is *irreflexive* iff $\forall a \in A, ((a, a) \notin R)$

Let us understand the definition of reflexivity in detail. Consider a set A and a binary relation $R \subseteq A \times A$, R is said to be a `reflexive relation` if for each $a \in A$, $(a, a) \in R$ and if there exists $a \in A$, $(a, a) \notin R$, then R is not reflexive. That is,

R is reflexive if and only if $\forall a (a, a) \in R$.

Note that the definition says, $[\forall a \in A,\ (a, a) \in R] \rightarrow R$ is reflexive and $[\exists a \in A,\ (a, a) \notin R] \rightarrow R$ is not reflexive. The first part of the definition is 'if part' of 'iff' and the contrapositive of second part of the definition is 'only if' of 'iff'.

Further, the definition of symmetricity says, if $(a, b) \in R$, then $(b, a) \in R$. Since, this is conditional, the empty relation is symmetric. That is, a relation not containing both (a, b) and (b, a). Similarly, for transitivity, if a relation contains just $(a, b) \in R$ but not (b, c), then R is transitive. Also, reflexivity and irreflexivity is defined with respect to the elements of the set A and all other properties are defined with respect to elements of $R \subseteq A \times A$.

Example 4. Let $A = \{1, 2, 3\}$. Identify which of the following relations defined on A are reflexive, symmetric and transitive.

$R_1 = \{(1, 1), (2, 2), (3, 3)\}$ $\quad R_2 = \{(1, 1), (1, 2), (1, 3)\}$
$R_3 = \phi$ $\quad R_4 = A \times A$
$R_5 = \{(2, 2), (3, 3), (1, 2)\}$ $\quad R_6 = \{(2, 3), (1, 2)\}$

Property	R_1	R_2	R_3	R_4	R_5	R_6
Reflexive	✓	X	X	✓	X	X
Symmetric	✓	X	✓	✓	X	X
Transitive	✓	✓	✓	✓	✓	X

1. R_2 is not reflexive as $(2, 2), (3, 3) \notin R_2$.
2. R_6 is not symmetric as $(3, 2), (2, 1) \notin R_6$.
3. R_6 is not transitive as $(1, 2), (2, 3) \in R_6$ but $(1, 3) \notin R_6$.

Example 5. $A = \{1, 2, 3, 4\}$. Identify the properties of relations.

$R_1 = \{(1, 1), (2, 2), (3, 3), (2, 1), (4, 3), (4, 1), (3, 2)\}$
$R_2 = A \times A, \quad R_3 = \phi, \quad R_4 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
$R_5 = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (4, 3), (3, 4)\}$

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Irreflexive	Transitive
R_1	×	×	×	✓	×	×
R_2	✓	✓	×	×	×	✓
R_3	×	✓	✓	✓	✓	✓
R_4	✓	✓	×	✓	×	✓
R_5	✓	✓	×	×	×	✓

Example 6. Let $R = \{(a, b) \mid a, b \in \mathbb{N} \text{ and } a \leq b\}$.
Since for all a in natural number set, $a \leq a$, $(a, a) \in R$. Therefore, R is reflexive. R is not symmetric as $1 \leq 2$ but not $2 \leq 1$. If $a \leq b$ and $b \leq c$, then it follows that $a \leq c$. Therefore, R is transitive. Since R is reflexive, R is not asymmetric. Since R does not contain both (a, b) and (b, a), $a \neq b$, R is antisymmetric. Since R is reflexive, R is not irreflexive.

Example 7. $R_7 = \{(a, b) \mid a, b \in \mathbb{I} \text{ and } a \text{ divides } b\}$.

Proof. Since 0 does not divide 0, $(0, 0) \notin R_7$. R_7 is not reflexive. For all $a, b, c \in \mathbb{I}$, if $(a, b) \in R$ and $(b, c) \in R$, then we prove that $(a, c) \in R$. Let $\frac{b}{a} = c_1 \geq 1$ and $\frac{c}{b} = c_2 \geq 1$. This implies $b = c_1 \cdot a$ and $c = c_2 \cdot b$. Therefore, $c = c_2 \cdot c_1 \cdot a = c_3 \cdot a$ where $c_3 = c_2 \cdot c_1$. Thus, $\frac{c}{a} = c_3 \geq 1$ and a divides c, $(a, c) \in R_7$. Note: R_7 is not symmetric as $(1, 3) \in R$ and $(3, 1) \notin R$. Also, it is antisymmetric, and it is not asymmetric and irreflexive □

Example 8. $R = \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a \text{ divides } b\}$.

Note $(0, 0) \notin R$, $(0.25, 0.75) \in R, (1, 4) \in R$. R is not reflexive. Also, $(1, x) \in R$, for any real number x but $(x, 1) \notin R$ for some x. Therefore, R is not symmetric. Using the argument given in Example 7, we can establish that R is transitive. Also, it is antisymmetric, and it is not asymmetric and irreflexive.

Remarks:
1. R is not reflexive does not imply R is irreflexive. Counter example: $A = \{1, 2, 3\}, R = \{(1, 1)\}$.

2. R is asymmetric implies that R is irreflexive. By definition, for all $a, b \in A, (a, b) \in R$ and $(b, a) \notin R$. This implies that for all $(a, b) \in R$, $a \neq b$. Thus, for all $a \in A$, $(a, a) \notin R$. Therefore, R is irreflexive.

3. R is not symmetric does not imply R is antisymmetric. Counter example: $A = \{1, 2, 3\}, R = \{(1, 2), (2, 3), (3, 2)\}$.

4. R is not symmetric does not imply R is asymmetric. Counter example: $A = \{1, 2, 3\}, R = \{(1, 2), (2, 2)\}$.

5. R is not antisymmetric does not imply R is symmetric. Counter example: $A = \{1, 2, 3\}, R = \{(1, 2), (2, 3), (3, 2)\}$.

6. R is reflexive implies that R is not asymmetric. By definition, for all $a \in A$, $(a, a) \in R$. This implies that, both (a, b) and (b, a) are in R when $a = b$. Thus, R is not asymmetric.

7. R is asymmetric and antisymmetric implies that R is transitive. Counter example: $A = \{1, 2, 3\}, R = \{(1, 2), (2, 3)\}$.

Counting Special Relations

We shall now count relations satisfying specific properties such as reflexivity, symmetricity, etc. Let $A = \{a_1, a_2, \ldots, a_n\}$ and we represent $A \times A$ as a matrix such that i^{th} row j^{th} column represents (a_i, a_j), $1 \leq i, j \leq n$. Observe that the matrix has n^2 elements.

Claim: The number of reflexive binary relations possible on A is $2^{n(n-1)}$.

Proof: By definition of reflexivity, observe that the diagonal elements of the matrix must be present in any reflexive binary relation. Therefore, the diagonal elements along with any subset from the remaining $n^2 - n$ elements is still a reflexive relation. So, the number of such sets is $2^{n(n-1)}$. Therefore, the number of reflexive binary relations is $2^{n(n-1)}$.

Claim: The number of symmetric binary relations possible on A is $2^{(n(n+1))/2}$.

Proof: Consider the elements other than the diagonal elements (off diagonal), we divide them into lower triangle elements ($i > j$) and upper triangle elements ($i < j$). Notice that in any symmetric relation, if there exists an element, say (a_i, a_j) from the lower triangle, then the element (a_j, a_i) from the upper triangle is also in the relation (this element is forced into the relation). Thus, if a subset from lower triangle element set is chosen, then its counter part from the upper triangle element set is also chosen ((a, b) is a counter part of (b, a), and vice versa). There are $n^2 - n$ such pairs of elements in lower triangle set and the number of subsets is $2^{(n^2-n)/2}$. Also, it is to be noted that any subset of the diagonal elements is symmetric, and together with a subset from lower triangle (or upper triangle) is also a symmetric relation. Therefore, the number of symmetric relations is $2^n \cdot 2^{(n^2-n)/2} = 2^{(n^2+n)/2}$.

`Approach 2`: Consider the matrix $A \times A$, let us find out how many elements in $A \times A$ are candidate elements for a symmetric relation. The candidate set must contain diagonal elements and either upper triangle or lower triangle elements. From Row 1, there are n elements and they are $\{(1,1),(1,2),\ldots,(1,n)\}$. From Row 2, it is $(n-1)$ and the set is $\{(2,2),(2,3),\ldots,(2,n)\}$. In general, the Row i contributes $(n-i+1)$ to the candidate set. Thus, the size of the candidate set is $n+n-1+\ldots+1 = \frac{n(n+1)}{2}$ elements, and any subset of the candidate element set is symmetric.

`Approach 3`: The candidate set for counting symmetric relations is $B = \{(a,a) \mid a \in A\} \cup \{(a,b) \mid a \neq b, a, b \in A\}$. The cardinality of B is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$. Any subset of B along with its counter part is a symmetric relation, and therefore, the number of symmetric binary relations possible in A is $2^{(n(n+1))/2}$.

Claim: The number of antisymmetric binary relations possible in A is $2^n \cdot 3^{(n^2-n)/2}$.

Proof: Consider an antisymmetric binary relation R and note that, if there exists an element, say (a_i, a_j) from the lower triangle of the matrix, then the element (a_j, a_i) from the upper triangle should not be present in R and vice versa. Therefore, there exists three possibilities for each (a_i, a_j) pair. That is, either (a_i, a_j) is in the relation, or (a_j, a_i) is in the relation, or none of $(a_i, a_j), (a_j, a_i)$ is in the relation. There are $(n^2 - n)/2$ pairs for (a_i, a_j) such that $i \neq j$. Therefore, there exists $3^{(n^2-n)/2}$ antisymmetric binary relations. Also, observe that any subset of the diagonal elements is also an antisymmetric relation. Therefore, the number of antisymmetric binary relations is $2^n \cdot 3^{(n^2-n)/2}$.

Claim: The number of binary relations on A which are both symmetric and antisymmetric is 2^n.

Proof: Suppose $(a_i, a_j), i \neq j$ is in the relation R, then due to symmetricity of R, (a_j, a_i) is also in the relation. However, this violates antisymmetric property. Therefore, for all $i \neq j$, (a_i, a_j) is not in the relation. Thus, the only possible elements for consideration are the diagonal elements. Observe that any subset of the diagonal elements is symmetric and antisymmetric. Therefore, the number of binary relations which are both symmetric and antisymmetric is 2^n.

Claim: The number of binary relations on A which are both symmetric and asymmetric is one.

Proof: Let R be a symmetric and asymmetric binary relation on any A. For all $a \in A$, none of the (a, a) elements is in the relation as R is asymmetric. Clearly, off diagonal elements (a, b), where $a \neq b$ are not present in R. Therefore, there does not exist any of the diagonal and off diagonal elements in R, and it follows that $R = \phi$, which is symmetric and asymmetric.

Claim: The number of binary relations which are both reflexive and antisymmetric in the set A is $3^{(n^2-n)/2}$.

Proof: Since all diagonal elements are part of the reflexive relation and there are 3 possibilities for each of the remaining $(n^2 - n)/2$ elements. Thus, we get $3^{(n^2-n)/2}$ binary relations which are reflexive and antisymmetric.

Claim: The number of asymmetric binary relations possible on the set A is $3^{(n^2-n)/2}$.

Proof: Similar to the argument for antisymmetric relations, note that there exists $3^{(n^2-n)/2}$ asymmetric binary relations, as none of the diagonal elements are part of any asymmetric binary relations.

Note:
1. If $A = \phi$ then $R = \phi$. R is reflexive and irreflexive. This is the only set and the relation having this property.

2. Counting transitive relations precisely is a challenging task. However, we can obtain good lower bounds and upper bounds. Any subset of the set of diagonal elements is an example transitive relation and thus, there are at least 2^n transitive relations (lower bound). A trivial upper bound is 2^{n^2}. To obtain a good upper bound, we focus on non-transitive relations. If we get a good lower bound for non-transitive relations, then total number of relations minus the lower bound gives a good upper bound for transitive relations. For example, irreflexive symmetric relations (except empty relation) are non-transitive relations, and there are at least $2^{\frac{n^2-n}{2}} - 1$ non-transitive relations. Therefore, a good upper bound for transitive relations is $2^{n^2} - 2^{\frac{n^2-n}{2}} + 1$.

Definitions:

R is an *Equivalence relation*, if R satisfies **R**eflexivity, **S**ymmetricity and **T**ransitivity. (RST properties)
R is a *Partial Order*, if R is **R**eflexive, **A**ntisymmetric and **T**ransitive. (RAT properties)

Operations on relations:

$R_1 - R_2 = \{(a, b) \mid (a, b) \in R_1 \text{ and } (a, b) \notin R_2\}$
$R_2 - R_1 = \{(a, b) \mid (a, b) \in R_2 \text{ and } (a, b) \notin R_1\}$

$R_1 \cup R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ or } (a,b) \in R_2\}$
$R_1 \cap R_2 = \{(a,b) \mid (a,b) \in R_1 \text{ and } (a,b) \in R_2\}$

Example 7. Let A be a set on integers and $R_1, R_2 \subseteq A \times A$ where $R_1 = \{(a,b) \mid a < b\}$ and $R_2 = \{(a,b) \mid a > b\}$. Check whether the following relations satisfy the properties of relations.

Note that $R_1 \cup R_2 = \{(a,b) \mid a < b \text{ or } a > b\}$. $R_1 \cap R_2 = \phi$

Relation	Reflexive	Symmetric	Asymmetric	Antisymmetric	Transitive
R_1	$\times$	$\times$	✓	✓	✓
R_2	$\times$	$\times$	✓	✓	✓
$R_1 \cup R_2$	$\times$	✓	$\times$	$\times$	$\times$
$R_1 \cap R_2$	$\times$	✓	✓	✓	✓
$R_1 - R_2$	$\times$	$\times$	✓	✓	✓
$R_2 - R_1$	$\times$	$\times$	✓	✓	✓

$R_1 \cup R_2$ is not transitive. Consider $(-1,2) \in R_1$ and $(2,-1) \in R_2$, and hence, $(-1,2) \in R_1 \cup R_2$ and $(2,-1) \in R_1 \cup R_2$. However, $(-1,-1) \notin R_1 \cup R_2$ as $-1 \not< -1$.
For the following theorems, we work with a set A and $R_1, R_2 \subseteq A \times A$.

Theorem 1. *If R_1 and R_2 are reflexive, and symmetric, then $R_1 \cup R_2$ is reflexive, and symmetric.*

Proof. Clearly, for each $a \in A$, there exists $(a,a) \in R_1$ and thus, $(a,a) \in R_1 \cup R_2$. Therefore, $R_1 \cup R_2$ is reflexive. We can claim that if $(a,b) \in R_1 \cup R_2$, then $(b,a) \in R_1 \cup R_2$. *Case 1:* if $(a,b) \in R_1$, then $(b,a) \in R_1$ as R_1 is symmetric and this implies that $(b,a) \in R_1 \cup R_2$. *Case 2:* if $(a,b) \in R_2$, then $(b,a) \in R_2$ as R_2 is symmetric and this implies that $(b,a) \in R_1 \cup R_2$. Therefore, we can conclude that $R_1 \cup R_2$ is symmetric and thus the theorem follows.

□

Remark: If R_1 is transitive and R_2 is transitive, then $R_1 \cup R_2$ need not be transitive.

Minimal counter example: Let $A = \{1,2\}$ such that $R_1 = \{(1,2)\}$ and $R_2 = \{(2,1)\}$. $R_1 \cup R_2 = \{(1,2),(2,1)\}$ and $(1,1) \notin R_1 \cup R_2$ implies that $R_1 \cup R_2$ is not transitive.

Note: From the above claim and theorem, it follows that if R_1 and R_2 are equivalence relations, then $R_1 \cup R_2$ need not be an equivalence relation. Is $R_1 \cap R_2$ an equivalence relation ?

Theorem 2. *If R_1 and R_2 are equivalence relations, then $R_1 \cap R_2$ is an equivalence relation.*

Proof. Clearly, for each $a \in A$, there exists $(a,a) \in R_1$ and $(a,a) \in R_1$. Therefore, for all $a \in A$, $(a,a) \in R_1 \cap R_2$. Therefore, $R_1 \cap R_2$ is reflexive. We now claim that if $(a,b) \in R_1 \cap R_2$, then $(b,a) \in R_1 \cap R_2$. Note that $(a,b) \in R_1$ and $(a,b) \in R_2$. Since R_1 and R_2 are symmetric $(b,a) \in R_1$ and $(b,a) \in R_2$. Therefore $(b,a) \in R_1 \cap R_2$. If $(a,b),(b,c) \in R_1 \cap R_2$, then $(a,b),(b,c) \in R_1$ and $(a,b),(b,c) \in R_2$. Since R_1 is transitive, $(a,c) \in R_1$. Similarly, since R_2 is transitive, $(a,c) \in R_2$. This implies that $(a,c) \in R_1 \cap R_2$. Therefore, we can conclude that $R_1 \cup R_2$ is transitive and thus, the theorem follows. □

Remark:
If R_1 and R_2 are equivalence relations on A,

1. $R_1 - R_2$ is not an equivalence relation (reflexivity fails).
2. $R_1 - R_2$ is not a partial order (since $R_1 - R_2$ is not reflexive).
3. $R_1 \oplus R_2 = R_1 \cup R_2 - (R_1 \cap R_2)$ is neither equivalence relation nor partial order (reflexivity fails)

Remark: If R_1 is antisymmetric and R_2 is antisymmetric, then $R_1 \cup R_2$ need not be antisymmetric.
Minimal counter example: Let $A = \{1, 2\}$ such that $R_1 = \{(1, 2)\}$ and $R_2 = \{(2, 1)\}$. $(1, 2), (2, 1) \in R_1 \cup R_2$ and $1 \neq 2$ implies that $R_1 \cup R_2$ is not antisymmetric.

Note: From the above claim it follows that if R_1, R_2 are partial order, then $R_1 \cup R_2$ need not be a partial order. Is $R_1 \cap R_2$ a partial order ?

Theorem 3. *If R_1 and R_2 are partial order, then $R_1 \cap R_2$ is a partial order.*

Proof. From the proof of Theorem 2, if R_1 and R_2 are reflexive, transitive, then $R_1 \cap R_2$ is reflexive, transitive. Now we shall show that if R_1 and R_2 are antisymmetric, then $R_1 \cap R_2$ is antisymmetric. On the contrary, assume that $R_1 \cap R_2$ is not antisymmetric. I.e., there exist $(a, b), (b, a) \in R_1 \cap R_2$ such that $a \neq b$. Note that $(a, b), (b, a) \in R_1$ and $(a, b), (b, a) \in R_2$ and it follows that R_1 and R_2 are not antisymmetric, which is a contradiction. Therefore, our assumption is wrong and $R_1 \cap R_2$ is antisymmetric. This implies that $R_1 \cap R_2$ is a partial order. This completes the proof of the theorem. $\square$

Questions:

1. Count the number of transitive relations in the set A.
2. Count the number of equivalence relations and partial ordered sets in the set A.
3. Prove using mathematical induction that $2^n \cdot 3^{(n^2-n)/2} \leq 2^{n^2}$.
4. Prove or disprove: If $A \neq \phi$ and $R \subseteq A \times A$, then R cannot be both reflexive and irreflexive. Give an example for which R is neither reflexive nor irreflexive.
5. Count the number of relations which are neither reflexive nor irreflexive.

Mysterious 22 [1]

Select any three-digit number with all digits different from one another. Write all possible two-digit numbers that can be formed from the three-digits selected earlier. Then divide their sum by the sum of the digits in the original three-digit number. See the result!!!

Composition of Relations

Let $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, Composition of R_2 on R_1, denoted as $R_1 \circ R_2$ or simply R_1R_2 is defined as $R_1 \circ R_2 = \{(a, c) \mid a \in A, c \in C \wedge \exists b \in B \text{ such that } ((a, b) \in R_1, (b, c) \in R_2)\}$.
Note: If $R_1 \subseteq A \times B$ and $R_2 \subseteq C \times D$, then $R_1 \circ R_2$ is undefined.
Let $R_1 \subseteq A \times B$, $R_2, R_3 \subseteq B \times C$, $R_4 \subseteq C \times D$.

Theorem 4. $R_1(R_2 \cup R_3) = R_1R_2 \cup R_1R_3$

Proof. Consider $(a,c) \in R_1(R_2 \cup R_3) \iff$ by definition, $\exists b \in B$ such that $(a,b) \in R_1 \wedge (b,c) \in R_2 \cup R_3$.
$\iff \exists b[(a,b) \in R_1 \wedge ((b,c) \in R_2 \vee (b,c) \in R_3)]$
$\iff \exists b[((a,b) \in R_1 \wedge (b,c) \in R_2) \vee ((a,b) \in R_1 \wedge (b,c) \in R_3)]$ (distribution law)
$\iff \exists b[((a,b) \in R_1 \wedge (b,c) \in R_2)] \vee \exists b[((a,b) \in R_1 \wedge (b,c) \in R_3)]$
$\iff (a,c) \in R_1R_2 \vee (a,c) \in R_1R_3 \iff (a,c) \in R_1R_2 \cup R_1R_3$ □

Theorem 5. $R_1(R_2 \cap R_3) \subset R_1R_2 \cap R_1R_3$

Proof. Let $(a,c) \in R_1(R_2 \cap R_3)$ by definition, $\exists b((a,b) \in R_1 \wedge (b,c) \in R_2 \cap R_3)$
$\iff \exists b[(a,b) \in R_1 \wedge (b,c) \in R_2 \wedge (b,c) \in R_3]$
$\iff \exists b[(a,b) \in R_1 \wedge (b,c) \in R_2 \wedge (a,b) \in R_1 \wedge (b,c) \in R_3]$
$\implies \exists b[(a,b) \in R_1 \wedge (b,c) \in R_2] \wedge \exists b[(a,b) \in R_1 \wedge (b,c) \in R_3]$
Note that, the biconditional operator is changed to implication as existential quantifier respects implication with respect to 'and' operator.
$\implies (a,c) \in R_1R_2 \wedge (a,c) \in R_1R_3 \implies (a,c) \in R_1R_2 \cap R_1R_3$ □

Theorem 6. $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, $R_3 \subseteq C \times D$. $(R_1R_2)R_3 = R_1(R_2R_3)$

Proof. Let $(a,d) \in (R_1R_2)R_3$ by definition, $\exists c((a,c) \in R_1R_2 \wedge (c,d) \in R_3)$
$\iff \exists c\ [\exists b[(a,b) \in R_1 \wedge (b,c) \in R_2] \wedge (c,d) \in R_3]$
$\iff \exists c\ \exists b\ [(a,b) \in R_1 \wedge (b,c) \in R_2 \wedge (c,d) \in R_3]$
$\iff \exists b\ \exists c\ [(a,b) \in R_1 \wedge (b,c) \in R_2 \wedge (c,d) \in R_3]$
$\iff \exists b\ [(a,b) \in R_1 \wedge \exists c\ [(b,c) \in R_2 \wedge (c,d) \in R_3]]$
$\iff \exists b\ [(a,b) \in R_1 \wedge (b,d) \in R_2R_3]$
$\iff (a,d) \in R_1(R_2R_3)$ □

Claim: If R_1 and R_2 are both reflexive, then $R_1 \circ R_2$ is reflexive.
Note that for all a, $(a,a) \in R_1$, and $(a,a) \in R_2$.
This implies that for all a, $(a,a) \in R_1 \circ R_2$.
□

Remarks:

Verify the following using examples.

- If R_1 and R_2 are both symmetric, then $R_1 \circ R_2$ need not be symmetric.
  $R_1 = \{(1,2),(2,1)\}$, $R_2 = \{(2,3),(3,2)\} \implies R_1 \circ R_2 = \{(1,3)\}$

- If R_1 and R_2 are both antisymmetric, then $R_1 \circ R_2$ need not be antisymmetric.
  $R_1 = \{(1,2),(3,4)\}$, $R_2 = \{(2,3),(4,1)\} \implies R_1 \circ R_2 = \{(1,3),(3,1)\}$

- If R_1 and R_2 are both transitive, then $R_1 \circ R_2$ need not be transitive.
  $R_1 = \{(1,2),(3,4)\}$, $R_2 = \{(2,3),(4,1)\} \implies R_1 \circ R_2 = \{(1,3),(3,1)\}$

- If R_1 and R_2 are both irreflexive, then $R_1 \circ R_2$ need not be irreflexive.
  $R_1 = \{(1,2)\}$, $R_2 = \{(2,1)\} \implies R_1 \circ R_2 = \{(1,1)\}$

Closure of a Relation

Let $R \subseteq A \times A$ be a non reflexive relation. To make R a reflexive relation, one can add the elements from $R' = (A \times A) \backslash R$ to R so that $R \cup R' (\subseteq A \times A)$ is reflexive. An interesting question is what will be the minimum cardinality of the set R' such that $R \cup R'$ is a reflexive relation. Closure operation deals with such minimum cardinality sets.

Let A be a finite set and $R \subseteq A \times A$.

Reflexive closure of R, denoted as $r(R)$ is a relation $R' \subseteq A \times A$ such that
(i) $R' \supseteq R$
(ii) R' is reflexive.
(iii) For any reflexive relation R'' $(\neq R')$ such that $R'' \supset R$, then $R'' \supset R'$.

Similarly, we can define symmetric closure $s(R)$ of R and transitive closure $t(R)$ of R.

Note:
$r(R) = R \cup E$ where $E = \{(x, x) \mid x \in A\}$.
$s(R) = R \cup R^c$ where $R^c = \{(a, b) \mid (b, a) \in R\}$.
$r(R)$ is a minimal superset of R which is reflexive and $s(R)$ is a minimal superset of R which is symmetric.

Example

$A = \{1, 2, 3\}$	r(R)	s(R)	t(R)
$R_1 = \{(1, 1), (2, 2), (3, 3)\}$	R_1	R_1	R_1
$R_2 = \{(1, 1), (2, 1)\}$	$\{(1, 1), (2, 1), (2, 2), (3, 3)\}$	$\{(1, 1), (2, 1), (1, 2)\}$	R_2
$R_3 = \phi$	$\{(1, 1), (2, 2), (3, 3)\}$	R_3	R_3
$R_4 = A \times A$	R_4	R_4	R_4
$R_5 = \{(1, 1), (2, 1), (2, 3)\}$	$\{(1, 1), (2, 1), (2, 3), (2, 2), (3, 3)\}$	$\{(1, 1), (2, 1), (2, 3), (1, 2), (3, 2)\}$	R_5

Some more examples
$A = \{1, 2, 3, 4\}$ $R_6 = \{(1, 2), (2, 1), (2, 3), (3, 4)\}$
$t(R_6) = \{(1, 2), (2, 1), (2, 3), (3, 4), (1, 1), (2, 4), (1, 4), (1, 3), (2, 2)\}$

$A = \{1, 2, 3, 4, 5\}$ $R_7 = \{(1, 2), (3, 4), (4, 5), (5, 3), (2, 1)\}$
$t(R_7) = \{(1, 2), (3, 4), (4, 5), (5, 3), (2, 1), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (4, 3), (3, 5), (5, 4)\}$

Relation as a graph

Note that each binary relation can be expressed as a directed graph. An example is illustrated below.
$A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 1), (2, 3), (3, 4)\}$

We define $R^2 = R \circ R$ and for all $i > 2$, R^i is composition of R on R^{i-1}. i.e., $R^i = R^{i-1} \circ R$.
$R^i = \{(a, b) \mid (a, c) \in R^{i-1} \wedge (c, b) \in R\}$
Using the above definition iteratively, we get

$R^2 = \{(1, 1), (2, 2), (1, 3), (2, 4)\}$

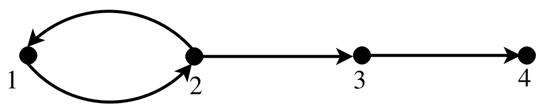


Fig. 1. Directed graph corresponding to R

$R^3 = \{(1,2),(2,1),(2,3),(1,4)\}$
$R^4 = \{(1,1),(2,2),(1,3),(2,4)\}$
$R^5 = \{(1,2),(2,1),(2,3),(1,4)\}$
Here $R_4 = R_2$, $R_5 = R_3$.
Note that R is not transitive, $R \cup R^2$ is also not transitive. However, $R \cup R^2 \cup R^3$ is transitive. This

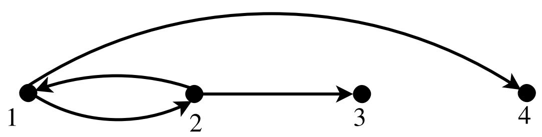


Fig. 2. Directed graph corresponding to R^3

is the smallest relation which is a superset of R and transitive. Therefore, $t(R) = R \cup R^2 \cup R^3$. Thus, $t(R)$ can be formulated as $t(R) =$ minimum i such that $R \cup \bigcup_{j=2}^{i} R^j$ is transitive. A natural question is to find out the upper bound on i.
Observe that for a finite set of size n there are at most 2^{n^2} distinct relations, therefore $t(R)$ in the worst case contain all 2^{n^2} relations which is $A \times A$. Otherwise, $t(R) \subset A \times A$.

Questions:

1. *Prove:* $(R_2 \cup R_3)R_4 = R_2R_4 \cup R_3R_4$
2. *Prove:* $(R_2 \cap R_3)R_4 \subset R_2R_4 \cap R_3R_4$

Amazing 1089 [1]

Select a non-palindrome 3 digit number xyz. Find their difference, say $abc = xyz - zyx$. See the value of $abc + cba$!!!

More on Transitive Closure

Given a relation R, a transitive closure of R is a minimal superset of R that is transitive. One approach to find $t(R)$ is to check whether $R^1 \cup R^2$ is transitive. If not, check $R^1 \cup R^2 \cup R^3$ is transitive and so on. It is certain that this approach terminates after some time. In fact, if the underlying directed graph is connected, then the longest path between any two nodes can not exceed n and due to which in the worst case $R^1 \cup R^2 \cup \ldots \cup R^n$ is transitive.
For the example given in Figure 1, $t(R) = R^1 \cup R^2 \cup R^3$. However, if the underlying directed graph is disconnected, then we observe the following. Consider the illustration given in Figure 3. $R^1 = \{(1,2),(2,1),(3,4),(4,5),(5,3)\}$. Note that the graph corresponding to R^1 is disconnected and has two components. It is a convention that R^0 denotes the pure reflexive relation. Transitive closure is $t(R) = R^1 \cup R^2 \cup R^3$, which is **max(m,n)** where m and n are the number of nodes in the two components.

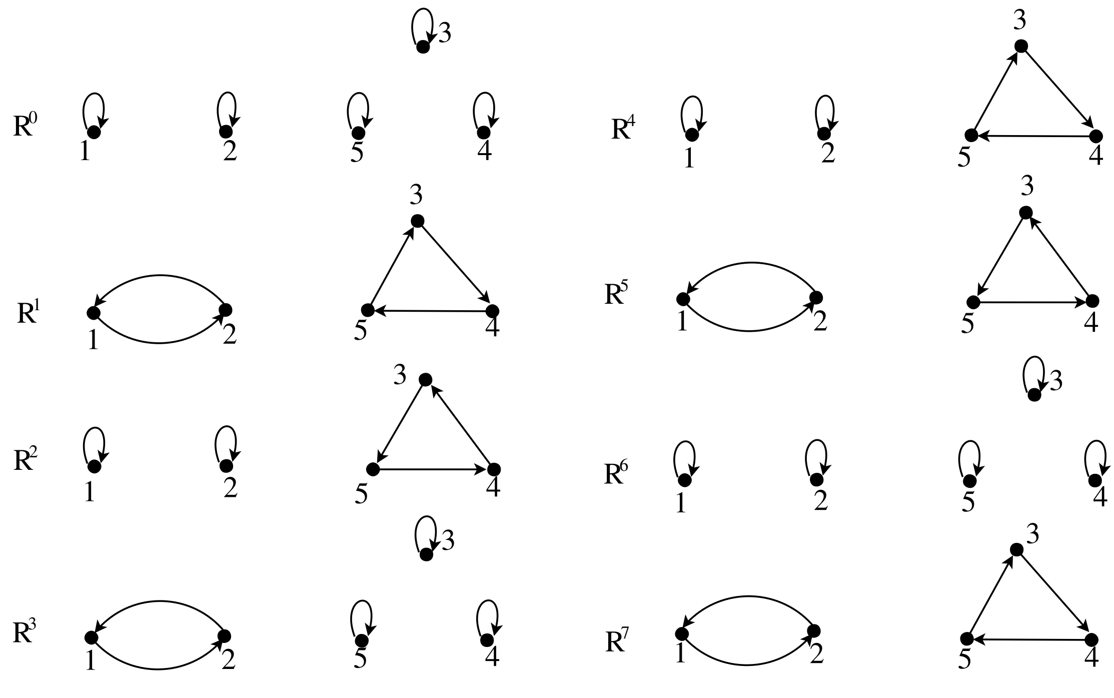


Fig. 3. R^1 to R^7 of $R = R^1$

Also, note that R^0 refers to equality relation or pure reflexive relation. Further, the smallest integers x, y such that $R^x = R^y$ in the graph is $x = 0$, and $y = 6$. In general, it is **LCM(m,n)**.

Remark: Let us consider the relation $R = \{(a,b) \mid b = a+1, a, b \in \mathbb{R}\}$. Note that the relation is infinite and $t(R) = \bigcup_{i=1}^{\infty} R^i$. Transitive closure of an infinite set is infinite.

Theorem 7. *Let R be an infinite relation.* $t(R) = \bigcup_{i=1}^{\infty} R^i$.

Proof. By definition $t(R)$ is transitive. Observe that if $(a,b) \in R^m$ and $(b,c) \in R^n$, then $(a,c) \in R^{m+n}$. To show that $\bigcup_{i=1}^{\infty} R^i$ is transitive, we use the above observation. Since $t(R)$ is minimal, transitive, and it contains R, $t(R)$ will be a subset of any transitive superset. I.e., R is such that $t(R) \supset R$ and $\bigcup_{i=1}^{\infty} R^i \supset R$, clearly $t(R) \subset \bigcup_{i=1}^{\infty} R^i$.

We prove $\bigcup_{i=1}^{\infty} R^i \subset t(R)$ by induction. *Base case:* $t(R) \supset R^1$. *Induction hypothesis:* Assume that for $k >= 1$, $t(R) \supset \bigcup_{i=1}^{k} R^i$. *Anchor step:* Let $(a,b) \in R^{k+1} \implies$ there exists c such that $(a,c) \in R^k$ and $(c,b) \in R$. Notice that $(a,c) \in t(R)$ from the induction hypothesis and $(c,b) \in t(R)$ from the base case and by transitivity, it follows that $(a,b) \in t(R)$. Therefore, $t(R) \supset \bigcup_{i=1}^{k} R^i$ for all $k \geq 1$. It can be concluded that $t(R) = \bigcup_{i=1}^{\infty} R^i$ □

Equivalence Class

We shall revisit equivalence relation in this section and introduce equivalence classes. We also explore paritition of a set and its connection to equivalence classes.

Consider the following equivalence relations defined on $A = \{1,2,3,4,5\}$.

$R_1 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$
$R_2 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,1),(1,2)\}$
$R_3 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(3,1),(2,4),(4,2)\}$
$R_4 = A \times A$

Definition: *Equivalence class* of $a \in A$ is defined as $[a]_R = \{x \mid (x,a) \in R\}$. If R is clear from the context, we drop the subscript R and shall denote the equivalence class of a as $[a]$

$[1]_{R_1} = \{1\}$	$[1]_{R_2} = \{1,2\}$	$[1]_{R_3} = \{1,3\}$
$[2]_{R_1} = \{2\}$	$[2]_{R_2} = \{1,2\}$	$[2]_{R_3} = \{2,4\}$
$[3]_{R_1} = \{3\}$	$[3]_{R_2} = \{3\}$	$[3]_{R_3} = \{1,3\}$
$[4]_{R_1} = \{4\}$	$[4]_{R_2} = \{4\}$	$[4]_{R_3} = \{2,4\}$
$[5]_{R_1} = \{5\}$	$[5]_{R_2} = \{5\}$	$[5]_{R_3} = \{5\}$

Remark: The notion equivalence class is defined for each $a \in A$, independent of whether R is an equivalence relation. For our discussion, we shall focus on R which are equivalence relations.
Properties: We consider an equivalence relation R defined on a set A.

1. $\bigcup_{\forall a \in A} [a] = A$
2. For every $a,b \in A$ such that $a \in [b]$, $a \neq b$, it follows that $[a] = [b]$.
3. $\sum_{\forall x \in A} |[x]| = |R|$.
4. For any two equivalence class $[a]$ and $[b]$, either $[a] = [b]$ or $[a] \cap [b] = \phi$.
5. For all $a,b \in A$, if $a \in [b]$ then $b \in [a]$.
6. For all $a,b,c \in A$, if $a \in [b]$ and $b \in [c]$, then $a \in [c]$.
7. For all $a \in A$, $[a] \neq \phi$.

Theorem 8. *If* $[a] \cap [b] \neq \phi$, *then* $[a] = [b]$

Proof. Given $[a] \cap [b] \neq \phi$. There exists $c \in [a] \cap [b]$. i.e., $c \in [a]$ and $c \in [b]$ and by definition $a \in [a]$, $b \in [b]$. We observe the following.

(1)	$(c, a), (c, b) \in R$	definition
(2)	$(a, c), (b, c) \in R$	(1) and symmetricity
(3)	$(a, b), (b, a) \in R$	(1), (2) and transitivity

Consider an arbitrary element $x \in [a]$. By definition $(x, a) \in R$. Since $(x, a) \in R$ and from (3) $(a, b) \in R$, by transitivity it follows that $(x, b) \in R$. This implies $x \in [b]$ and thus $[a] \subseteq [b]$. Similarly, consider an arbitrary element $y \in [b]$. By definition $(y, b) \in R$. Since $(y, b) \in R$ and from (3) $(b, a) \in R$, by transitivity it follows that $(y, a) \in R$. This implies $y \in [a]$ and thus $[b] \subseteq [a]$. Therefore, we conclude $[a] = [b]$. □

Theorem 9. $\bigcup_{x \in A} [x] = A$

Proof. Let $y \in \bigcup_{x \in A} [x]$. Clearly, $y \in [y]$ and since $[y] \subseteq A$, $y \in A$. This implies $\bigcup_{x \in A} [x] \subset A$. Consider $y \in A$. $y \in [c]$ for some $c \in A$. Since $[c] \subset \bigcup_{x \in A} [x], y \in \bigcup_{x \in A} [x]$ and it follows that $A \subset \bigcup_{x \in A} [x]$. Therefore we conclude $\bigcup_{x \in A} [x] = A$. □

Remarks:

1. The proof of the theorem 'If $[a] \cap [b] \neq \phi$, then $[a] = [b]$' does not make use of the property of R being reflexive, and hence the theorem is overstated. That is, with symmetricity and transitivity, one can claim $[a] = [b]$. Thus, the claim 'If R is symmetric and transitive, then for any two equivalence classes $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \phi$.

2. Property 5 is true as R is symmetric. By definition, $a \in [b]$ implies that $(a, b) \in R$. Since R is symmetric, $(b, a) \in R$, and therefore, $b \in [a]$.

3. Property 6 is true as R is transitive. By definition, $(a, b), (b, c) \in R$. Since R is transitive, $(a, c) \in R$ and therefore, $a \in [c]$.

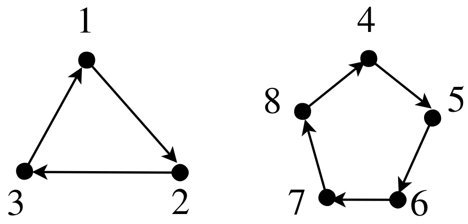
4. Property 7 is true as R is reflexive. $a \in [a]$ and therefore, $[a] \neq \phi$

5. Proof of Property 2: Suppose, there exists $c \in [a]$ and $c \notin [b]$. By definition, $(c, a) \in R$. Since $a \in [b]$, $(c, b) \in R$ and hence, $c \in [b]$. This is a contradiction. Therefore, $[a] = [b]$.

Definition: *Clique* is a completely connected subgraph of a graph. Given an equivalence relation and its directed graph representation, we observe that the vertices corresponding to equivalence classes induce a clique.

Questions:

1. *Prove using counting technique and PHP (Pigeon hole principle) or use mathematical induction:* Let R be a finite relation and G be the directed graph corresponding to R. $t(R) = \bigcup_{i=1}^{n} R^i$, where $R^i = R^{i-1}R$, $1 \leq i \leq n$ and n is the length of longest path in G.
2. Find R^1 to R^{16} for the figure shown below.



Counting Equivalence Relations

In this section, we shall ask; How many equivalence relations are possible on a set A?

MATH WONDERS [1]

Find the digit corresponding to each letter

S E N D
+ M O R E
= M O N E Y

Definition:
For a set A, the partition of A is $\{A_1, \ldots, A_k\}$ such that
(i) Each $A_i \subseteq A$
(ii) For any two A_i, A_j, $A_i \cap A_j = \emptyset$
(iii) $\cup_{i=1}^{i=k} A_i = A$.
For example, $A = \{1, 2, 3, 4\}$, a partition of A is $\{\{1, 2\}, \{3\}, \{4\}\}$. Another example of partition of A is $\{\{1\}, \{2, 3, 4\}\}$.

Observations:
1. The number of equivalence relations on A is same as the number of ways of listing all possible sets of equivalence classes of A
2. Each set of equivalence classes corresponds to a partition of the set A.
3. Listing all possible sets of equivalence classes of A is same as listing all partitions of A.
4. Counting the number of equivalence relations on A is equivalent to counting the number of partitions of A.

Consider the set $A = \{1, 2, 3\}$, the possible equivalence relations, equivalence classes and partition are listed below;

Equivalence Relation	Equivalence Class	Partition
$R_1 = \{(1,1), (2,2), (3,3)\}$	$[1] = \{1\}, [2] = \{2\}, [3] = \{3\}$	$\{\{1\}, \{2\}, \{3\}\}$
$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$	$[1] = \{1,2\}, [2] = \{1,2\}, [3] = \{3\}$	$\{\{1,2\}, \{3\}\}$
$R_3 = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$	$[1] = \{1,3\}, [2] = \{2\}, [3] = \{1,3\}$	$\{\{1,3\}, \{2\}\}$
$R_4 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$	$[1] = \{1\}, [2] = \{2,3\}, [3] = \{2,3\}$	$\{\{2,3\}, \{1\}\}$
$R_5 = A \times A$	$[1] = \{1,2,3\}, [2] = \{1,2,3\}, [3] = \{1,2,3\}$	$\{\{1,2,3\}\}$

If $A = \{1, 2\}$, then there are two sets of equivalence classes; (i) $[1] = \{1\}, [2] = \{2\}$ (ii) $[1] = [2] = \{1, 2\}$, and the associated partitions are (i) $\{\{1\}, \{2\}\}$, (ii) $\{\{1, 2\}\}$. One can generate the partitions of $\{1, 2, 3\}$ from $\{1, 2\}$. Consider the element $'3'$.
From the partition $\{\{1\}, \{2\}\}$, we obtain the partitions $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}\}$ and from the partition $\{\{1, 2\}\}$, we obtain the partitions $\{\{1, 2\}, \{3\}\}$, $\{\{1, 2, 3\}\}$. Thus, all five partitions of the set $\{1, 2, 3\}$ can be obtained from the set $\{1, 2\}$. Since each partition corresponds to an equivalence relation, we generate all equivalence relations of the set $\{1, 2, 3\}$ using this approach.
This approach also indicates that one can obtain a recursive formula to obtain all partitions of

the set $\{1, 2, \ldots, n\}$. The following counting argument was discoverd by the mathematician Bell. Let B_n be the number of partitions (equivalence relations) possible on a set A of n elements. We now present a recurrence relation which will count the number B_n. The approach is to generate all partitions using a specific partition of the same set under consideration. Consider a partition of n elements, say for example, $P = \{\{1, 2\}, \{3, 4, 5\}, \{6, \ldots, n\}\}$. Using this partition, we recursively generate all other partitions.
In general $P = \{A_1, A_2, \ldots, A_p\}$ and $A_i \subseteq \{1, \ldots, n\}$ and each A_i is distinct. In P, consider A_i containing the element n and assume that $k = |A_i \setminus \{n\}|$. These k elements of A_i can be any subset in $\{1, \ldots, n-1\}$. Therefore, the number of such possible sets for A_i is $\binom{n-1}{k}, k \in \{0, \ldots, n-1\}$. Now to establish a recursive relation we focus on the remaining $n - k - 1$ elements which are distributed among $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$. Interestingly, there are B_{n-k-1} partitions among $n - k - 1$ elements. Thus, we get $B_n = \sum\limits_{k=0}^{n-1} \binom{n-1}{k} B_{n-k-1}$.

Note: $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$.
$B_5 = \binom{4}{0}B_4 + \binom{4}{1}B_3 + \binom{4}{2}B_2 + \binom{4}{3}B_1 + \binom{4}{4}B_0 = 15 + 20 + 12 + 4 + 1 = 52$.
The number B_n is known as the Bell's number in the literature.

Rank of an equivalence relation is the number of distinct equivalence classes.

For example, consider the relations R_1 to R_4 given below, their ranks are;
$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$
$R_2 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (2, 1), (1, 2)\}$
$R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (2, 4), (4, 2)\}$
$R_4 = A \times A$

Relation	Rank
R_1	5
R_2	4
R_3	3
R_4	1

In general, $A \times A$ has the least rank (one) and the pure reflexive relation (equality relation) has the highest rank (n).

Revisit: Partial Order

In this section, we shall revisit partial order and discuss some more properties and special relations in detail. Recall, a relation R is a partial order if it satisfies reflexivity, antisymmetricity, and transitivity. Further, R is a *Quasi order* if R is transitive and irreflexive.
Trichotomy Property: For all elements $a, b \in A$, exactly one of the following holds:
(i). $(a, b) \in R$. (ii). $(b, a) \in R$. (iii). $(a = b)$.
A binary relation is a *total order* if it is a partial order and satisfies the trichotomy property. $R = \{(a, b) \mid a, b \in \mathbb{N}, a \leq b\}$ is a partial order as well as total order.

Graphical Representation of Partial order relations: Hasse Diagram
Consider a partial order R and the associated graphical representation G of R. In G, we make the following changes to obtain a graph H. The changes are (i) remove reflexivity arcs (ii) remove transitivity arcs (iii) make antisymmetric arcs undirected. The resultant graph is known as Hasse diagram. An interesting observation is that Hasse diagram (diagram representing just antisymmetric arcs) brings an ordering (hierarchy) among elements. That is, arcs such as $(1,2),(1,3),(2,3)$ can be represented as 2 above 1 and 3 above 2, and the transitive arc $(1,3)$ is ignored.

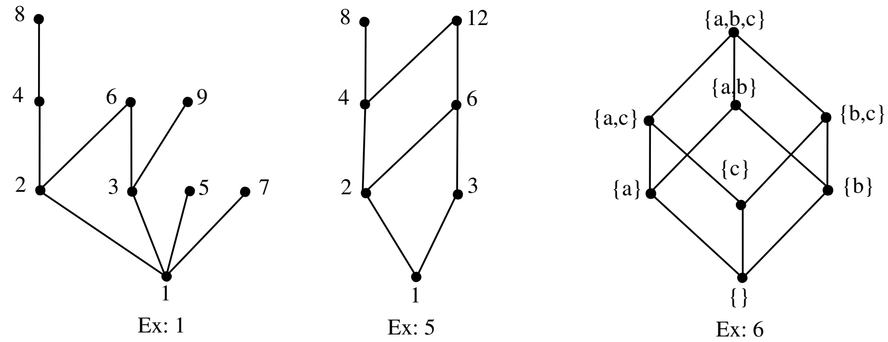


Fig. 4. Hasse diagram of Examples 1,5 and 6

Example 1: Consider the relation R: a divides b on the set $A = \{1, 2, \ldots, 9\}$.
Example 5: Let $A = \{1, 2, \ldots, 6\}$ and $R = \{(a, b) \mid a \text{ divides } b\}$
Example 6: $R = \{(A, B) \mid A \subseteq B\}$
In all of the above examples, we see that there is an ordering among elements of A. The arc (x, y) is represented by positioning y above x.
Example 1: Consider the relation R: a divides b on the set $A = \{1, 2, \ldots, 9\}$. R is a partial order and not a total order as the elements $(3, 5), (6, 9) \notin R$.
Example 2: $(Z, \geq)$ is a partial order and a total order.
Example 3: $(Z, >)$ is a not a partial order as reflexivity fails.
Example 4: $(Z^+, |)$ ($|$ denotes 'divides') is a partial order but not a total order.
Example 5: Let $A = \{1, 2, \ldots, 6\}$ and $R = \{(a, b) \mid a \text{ divides } b\}$ is a partial order but not a total order as $(3, 8), (8, 3) \notin R$
Example 6: $R = \{(A, B) \mid A \subseteq B\}$ on the power set of $\{a, b, c\}$ is a partial order but not a total order as $(\{a, c\}, \{b, c\}), (\{b, c\}, \{a, c\}) \notin R$

Question 1 Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 3), (1, 3)\}$. Find a minimum augmentation relation R' such that $R \cup R'$ is a total order.
Ans: $R' = \{(3, 4), (2, 4), (1, 4)\}$
Question 2 Let $A = \{1, 2\}$, $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ and $R = \{(A, B) \mid A \subseteq B, A, B \in P(A)\}$. This is a partial order and can not be converted into a total order by augmenting minimum pairs. Not all partial orders can be converted into a total order.

Since there is an ordering among elements of a poset, one can identify special elements in a poset.
Definition:
Let $(A, \preceq)$ be a poset and $B \subseteq A$. ($\preceq$ means some relation)

1. An element $b \in B$ is the *greatest element* of B if for every $b' \in B, b' \preceq b$.
2. An element $b \in B$ is the *least element* of B if for every $b' \in B, b \preceq b'$.

For Example 6,

Set B	greatest element	least element
$\{\{a\},\{a,c\}\}$	$\{a,c\}$	$\{a\}$
$P(\{a,b,c\})$	$\{a,b,c\}$	$\{\}$
$\{\{a\},\{b\},\{c\},\{\}\}$	NIL	$\{\}$
$\{\{a\},\{b\},\{c\},\{a,b\}\}$	NIL	NIL
$\{\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$	$\{a,b,c\}$	NIL

Theorem 10. *The greatest and least elements in a poset are unique.*

Proof. Let $R \subset A \times A$ be a partial order relation. On the contrary, assume that there exist two greatest elements $g_1, g_2 \in A$ such that $g_1 \neq g_2$. Since g_1 is a greatest element, $g_1 \preceq g, \forall g \in A$ and it follows that $g_1 \preceq g_2$. Similarly, since g_2 is a greatest element, $g_2 \preceq g_1$. It follows that $(g_1, g_2) \in R$ and $(g_2, g_1) \in R$. This contradicts the fact that R is antisymmetric. Similar argument can be made for the fact that there exists a unique least element.

□

Definition: $(A, \preceq)$ is a *well order* if $(A, \preceq)$ is a total order and for all $A' \subseteq A, A' \neq \phi, A'$ has a least element.

Note: Every finite totally ordered set is well ordered. For a finite set which is a partial order and total order, clearly, for any non-empty subset, the least element exists. Observe that the Hasse diagram of a total order is a path graph (linear chain) on n nodes. Any non-empty subset of a total order is equivalent to any sub-path in the Hasse diagram which has the least element.

Example:

Relation	Total order	Well-order
$(\mathbb{R}, \leq)$	✓	×
$(\mathbb{N}, \leq)$	✓	✓
$(\mathbb{I}, \leq)$	✓	×

Remark: If we consider a subset $A' \subseteq A$ such that $A' = \mathbb{I}$ or $A' = \mathbb{I}^-$, then $(A', \leq)$ does not have a least element. Also, there does not exist a least element for $(\mathbb{R}^+, \leq)$.

Note: Mathematical induction can be applied only to well ordered sets as there is a least element for every non-empty subset which implicitly gives an ordering among elements. This shows that, proving claims on well-ordered sets using mathematical induction proof technique is appropriate. Further, for well-ordered sets, the successor is well defined for each element. For sets with no proper definition for successor of an element (for example, real numbers), the mathematical induction can not be used to prove claims on such sets.

Remark: Consider a set A such that $|A| = n$. Since each total order on A is a path (linear chain), the number of total orders possible on A is the number of path like Hasse diagrams on n nodes. Note that this is equal to the number of permutations on n nodes, which is $n!$. i.e., # total orders = # path like Hasse diagrams = # permutations = $n!$

Special Elements in a poset

Definition: Let $(A, \preceq)$ be a poset, $B \subseteq A$

1. An element $b \in B$ is a maximal element of B if $b \in B$ and there does not exist $b' \in B$ such that $b \neq b'$ and $b \preceq b'$. Similarly, minimal elements of B can be defined.
2. An element $b \in A$ is upper bound for B if for every element $b' \in B$, $b' \preceq b$. Similarly, lower bound of B can be defined.
3. An element $b \in A$ is a least upper bound (lub) for B if b is an upper bound and for every upper bound b' of B, $b \preceq b'$.
4. An element $b \in A$ is a greatest lower bound (glb) for B if b is a lower bound and for every lower bound b' of B, $b' \preceq b$.

For Figure 4 associated with Example 5

Set B	minimal elements	maximal elements	Lower bound	Upper bound
A	$\{1\}$	$\{8, 12\}$	$\{1\}$	NIL
$\{2, 3, 4\}$	$\{2, 3\}$	$\{3, 4\}$	$\{1\}$	$\{12\}$

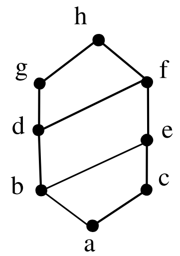


Fig. 5. Hasse diagram 2

For Figure 5

Set B	minimal elements	maximal elements	Lower bound	Upper bound	LUB	GLB
A	$\{a\}$	$\{h\}$	$\{a\}$	$\{h\}$	$\{h\}$	$\{a\}$
$\{b, c, d, e\}$	$\{b, c\}$	$\{d, e\}$	$\{a\}$	$\{f, h\}$	$\{f\}$	$\{a\}$
$\{a, b, c\}$	$\{a\}$	$\{b, c\}$	$\{a\}$	$\{e, f, h\}$	$\{e\}$	$\{a\}$
$\{d, e\}$	$\{d, e\}$	$\{d, e\}$	$\{b, a\}$	$\{f, h\}$	$\{f\}$	$\{b\}$

Lexicographic and Standard orderings

In this section, we introduce an ordering among elements of a set.
We first introduce lexicographic ordering:

Given $\sum$: finite alphabet, for example $\sum = \{a, b\}$

If $x, y \in \sum^*$, then $x \leq y$ in the lexicographic ordering
(x precedes y) of $\sum^*$ if
(i) x is a prefix of y (or)

(ii) $x = zu$ and $y = zv$ where $z \in \sum^*$ is the longest prefix common to x and y and u precedes v in the lexicographic ordering.

$\sum = \{a\}$ $\sum^* = \{a, aa, aaa, \ldots\}$ is a partial, total and well ordered set.
$\sum = \{a, b\}$ $\sum^* = \{a, aa, aaa, \ldots, aa\ldots ab, aa\ldots ba, \ldots, b, ba, baa, \ldots\}$ is a partial, and total ordered set but not a well ordered. For example, there is no least element for the set $B = \{b, ab, aab, aaaab, aa\ldots ab\}$.

Standard ordering
Notation: $\sum$: alphabet and $\sum^*$ is the set of all strings over $\sum$. $||x||$ is the length of string $x \in \sum^*$
Definition:
$x \leq y$ if
(i) $||x|| < ||y||$ or
(ii) $||x|| = ||y||$ and x precedes y in the lexicographic ordering of $\sum^*$

Note that standard ordering is a poset, total and well ordered set as we can order $\sum^*$ as $(a, b,\ aa, ab, ba, bb,\ aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots)$

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