Practice Questions - Infinite Sets

1. Determine which of the following sets are finite and which are infinite.
   (a) The set of all strings $\{a, b, c\}^*$ of length no greater than k.
   Finite set (Since k is fixed). Number of strings of length zero = 1. Number of strings of length one = 3. Number of strings of length two = $3^2 = 9$. Thus, number of strings of length no greater than k is $3^0 + 3^1 + \ldots + 3^k$.

   (b) The set of all m x n matrices with entries from $\{0, 1, \ldots, k\}$ where k, m, n are positive integers.
   Number of entries in a $m \times n$ matrix is mn and each entry has $k+1$ options, thus there are $(k+1)^{mn}$ possible $m \times n$ matrices if k, m and n are fixed, which is a finite set. If k is a variable and m, n are fixed, then the number of possibilities are $\sum_{k=1}^{\infty}(k+1)^{mn}$. This is equivalent to enumerating all integer arrays of size mn, which is a countably infinite set.

   (c) The set of all functions from {0, 1} to **I**.
   Countably infinite. Since, $f(0) = a$ and $f(1) = b$, where $(a, b) \in I \times I$, which is countably infinite.

   (d) The set of all polynomials of degree two with integer coefficients.
   Countably infinite. Since, counting the set of all polynomials with integer co-efficients is equivalent to count $I \times I \times I$. i.e., $\{(a_1, a_2, a_3) \mid a_i$ is a co-efficient $\}$ and the number of such (a_1, a_2, a_3) is $I \times I \times I$

2. Prove that the intersection of two infinite sets is not necessarily infinite. (the class of infinite sets is not closed under intersection) Let $A = I^+ \cup \{0\}$ and $B = I^+ \cup \{0\}$. Clearly, A and B are infinite sets. $A \cap B = \{0\}$, which is finite.

3. Let A and B be infinite sets such that $B \subset A$. Is the set $A - B$ necessarily finite? Is it necessarily infinite? Give examples to support your answer. It can be either finite or infinite. For example: If $A = I$ and $B = N$ are the infinite sets then $A - B$ is infinite. If $A = N$ and $B = I^+$ are the infinite sets then $A - B$ is finite.

4. Check whether the following sets are countable.(countably finite or countably infinite)
   (a) The set of all functions from the set $\{0, 1, \ldots, k-1\}$ to **N**, where k is a fixed integer. $f : \{0, 1, \ldots, k-1\} \to N$ such that $f(0) = a_1, f(1) = a_2$, ..., $f(k-1) = a_k$, where $a_i \in N$. Thus the problem is equivalent to finding the number of elements in $N \times N \times \ldots \times N (k$ times), which is countably infinite.

   (b) Let S be a countably infinite set. (i) Set of all subsets of **S** (ii) set of all finite subsets of **S**.
   - (i): Since the power set of the natural number set is uncountable, the set of all subsets is uncountable.
   - (ii): Let k be the size of largest finite subset. A_1: set of singleton sets, A_i: set of i-element sets, $1 \leq i \leq k$. The set A_i can be seen as $(N \times N \times \ldots \times N)$ (i times)

which is a countably infinite set. The required set is $\cup_{i=1}^{k} A_i$, since countable union of countably infinite sets is countably infinite, the result is countably infinite.

(c) The set of computational problems in computer science. Uncountable, for example; $P_{ij}=$ print the open interval (i, j) where $i, j \in R$. The number of such P_{ij}'s is the cardinality of real numbers and hence uncountable.

(d) $A = \{(x_1, \ldots, x_k) \mid x_i \in \mathbf{N}$ and k is a fixed integer $\}$ Countably infinite as $N \times N \times \ldots \times N$($k$ times) is countably infinite.

(e) $A = \{(x_1, \ldots, x_k) \mid x_i \in \mathbf{N}$ and k is a variable integer $\}$ Since each element (vector of k elements) is a subset of N, the number of such vectors is the cardinality of the power set of N. Therefore, uncountable.