Practice Questions - Relations and Proof Techniques

1. Prove that $2^0 = 1$.
Solution:
$2^0 = 2^{x-x} = \frac{2^x}{2^x} = 1$

2. What is the powerset of $\{1, \{2\}, \emptyset, \{\emptyset\}\}$.
Solution:
$\{\emptyset,$
$\{1\}, \{\{2\}\}, \{\emptyset\}, \{\{\emptyset\}\},$
$\{1, \{2\}\}, \{1, \emptyset\}, \{1, \{\emptyset\}\}, \{\{2\}, \emptyset\}, \{\{2\}, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\},$
$\{1, \{2\}, \emptyset\}, \{1, \{2\}, \{\emptyset\}\}, \{2, \emptyset, \{\emptyset\}\}, \{1, \emptyset, \{\emptyset\}\},$
$\{1, \{2\}, \emptyset, \{\emptyset\}\}\}$

3. Prove that the empty set is a subset of every set.
Solution:
Definition of a subset: $A \subseteq B$ iff $\forall x(x \in A \rightarrow x \in B)$. Note that $\forall x P(x)$ is true if UOD is empty (we need at least one element in UOD to disprove) and $\exists x P(x)$ is false if UOD is empty (we need at least one element in UOD to prove). If A is empty (UOD is empty), then the premise of sufficiency is false and the claim is true. This is called 'trivial' proof or the statement is 'vacuously' true.

4. Show that $\sqrt{3}$ is irrational.
Solution:
$\sqrt{3} = \frac{a}{b}$; let the representation be in simplified form, i.e., $gcd(a, b) = 1$
$\Rightarrow 3 = \frac{a^2}{b^2}$
$\Rightarrow a^2 = 3 \cdot b^2$
a^2 is a multiple of 3 and hence, a is also a multiple of 3. If a is not a multiple of 3, then a is of the form either $3k + 1$ or $3k + 2$. Thus, a^2 is of the form either $9k^2 + 6k + 1$ or $9k^2 + 12k + 4$, which is a contradiction to our assumption that a^2 is a multiple of 3. So, assume that $a = 3k$, for some $k \in \mathbf{N}$. Thus,
$$a^2 = 3 \cdot b^2 \Rightarrow 9k^2 = 3b^2 \Rightarrow 3k^2 = b^2 \Rightarrow b = \pm 3k$$
Hence, b is also a multiple of 3. In this case $\frac{a}{b}$ is not in simplest form ($gcd(a, b) \geq 3$), which is a contradiction.

5. You are given a box of size $\sqrt{3} \times \sqrt{3} \times \sqrt{3}$. Is the size of the box finite or infinite. Justify.
Solution:
Consider a box of size $3 \times 3 \times 3$ (unit: say, m^3). Clearly, the box is finite and it contains any box of size $\sqrt{3} \times \sqrt{3} \times \sqrt{3}$. Any sub box of a finite box is also finite. (any subset of a finite set is finite)

6. Present a direct proof: $2^n \leq n! \leq n^n$.
Solution:
$2^n = 2 \times 2 \times \ldots \times 2$; 2 appearing n times.
Clearly, $2 \times 2 \times \ldots \times 2 \leq n \times (n-1) \times \ldots \times 2 \times 1$. Note that
$2 \not\leq 1$, $2 \times 2 \not\leq 2 \times 1$, $2 \times 2 \times 2 \not\leq 3 \times 2 \times 1$, whereas, $2 \times 2 \times 2 \times 2 \leq 4 \times 3 \times 2 \times 1$. That is, $2 \times 2 \times 2 \times 2 \leq 4 \times 3 \times 2 \times 1$ is equivalent to $2 \times 2 \times 2 \times 2 \leq 2 \times 2 \times 3 \times 2 \times 1$. Further, $2 \times 2 \times 2 \times 2 \times 2 \leq 5 \times 4(2 \times 2) \times 3 \times 2 \times 1$. Thus, for $n \geq 4$, $2^n \leq n!$.

Further, $n \times (n-1) \times \ldots \times 2 \times 1 \leq n \times n \times \ldots \times n$, for $n \geq 1$. Thus, the claim follows for $n \geq 4$.

7. Suppose that the 10 integers $1, 2, \ldots, 10$ are randomly positioned around a circular wheel. Show that the sum of some set of 3 consecutively positioned numbers is at least 17.
Solution:

Let a_i denote a label in the range [1..10]. On the contrary, assume that the sum of any three consecutive sectors is ≤ 16. Therefore,
$a_1 + a_2 + a_3 \leq 16$
$a_2 + a_3 + a_4 \leq 16$
.
.
$a_{10} + a_1 + a_2 \leq 16$

$$\begin{aligned} \sum_{i=1}^{10} a_i &\leq 10 \times 16 \\ 3 \times (1 + 2 + \ldots + 10) &\leq (10 \times 16) \\ 165 &\leq 160 \ \ (\texttt{a contradiction}) \end{aligned}$$

Thus, our assumption that the sum of any three consecutive sectors is ≤ 16 is wrong. Therefore, there exist 3 consecutive sectors such that the sum of their assigned numbers is at least 17.

8. For each positive integer n, there are more than n prime integers.
Solution:
Base case: $n = 1$. $\{2, 3, \ldots, \}$ are prime integers. Clearly, for $n = 1$, there exists more than one.
Induction hypothesis: Assume for $n = k, k \geq 1$, that there exists more than k prime integers. Let the prime numbers be $p_1, p_2, \ldots, p_k, p_{k+1}, \ldots$.
Induction step: We claim that for $n = k + 1$, $k \geq 1$ there exists more than $k + 1$ prime numbers. Consider the number $P = p_1 \cdot p_2 \ldots p_k \cdot p_{k+1} + 1$, i.e., P is one plus the product of the prime numbers $p_1, p_2, \cdots, p_{k+1}$.
We consider the following cases to complete the proof.
Case a: If P is a prime number, then there exists more than $k + 1$ prime numbers with $(k + 2)^{nd}$ prime number being P.
i.e., $\{p_1, p_2, \ldots, p_k, p_{k+1}, P\}$ are the set of $(k + 2)$ prime numbers.
Case b: If P is not a prime number, then note that there exists a prime factorization for P and none of $\{p_1, p_2, \ldots, p_k, p_{k+1}\}$ are its prime factors. This implies that there exists a prime factor p_{k+2} for P such that $p_{k+2} \neq p_i$, $1 \leq i \leq k + 1$. Therefore, $\{p_1, p_2, \ldots, p_k, p_{k+1}, p_{k+2}\}$ are prime numbers with cardinality more than $k + 1$. The induction is complete and hence the claim follows.

9. Show that $F_n \leq (\frac{12}{7})^n$, where F_n is the n^{th} Fibonacci number.
Solution:
`Base:` $n = 1$, $F_1 = 1 \leq (\frac{12}{7})^1$, $F_2 = 1 \leq (\frac{12}{7})^2$.
`Strong induction hypothesis:` Assume the claim is true for F_{k-1} and $F_k, k \geq 2$. Note that, two base cases must be proved as the hypothesis make assumption about F_{k-1} and F_k. As part of the hypothesis, we assume that $F_k \leq (\frac{12}{7})^k$ and $F_{k-1} \leq (\frac{12}{7})^{k-1}$. `Induction Step:` Consider $F_{k+1} = F_k + F_{k-1}$. $F_{k+1} \leq (\frac{12}{7})^k + (\frac{12}{7})^{k-1} = (\frac{12}{7})^k(1 + \frac{7}{12}) < (\frac{12}{7})^{k+1}$. Thus, the claim follows for $n \geq 1$.

10. Recall currency change problem. Given the denominations R_5 and R_9, show that change for R_x can be given using these two denominations.
Solution:
Let us prove this by induction on n. **Base Case:** $n = 35$. Seven \$5's.
Hypothesis: $n = k, k \geq 35$. Assume that \$$k$ request can be served using \$5 and \$9.
Induction Step: $n = k + 1, k \geq 35$. We will divide this into two cases
Case 1: There exist at least one \$9.
Replace one \$9 with two \$5.
Case 2: There exist at least seven \$5.
Replace seven \$5 with four \$9.

The induction is complete and hence the claim follows. Note: Induction works fine even if we assume base case to be $n = 32$. In fact any $n \geq 32$ works fine.

11. Show that $(11)^{n+2} + (12)^{2n+1}$ is divisble by 133.
Solution:
Base Case: $n = 0$. $11^2 + 12^1 = 133$, which is divisible by 133.
Hypothesis: Assume that the statement is true for $n = k$, $k \geq 0$.
i.e., $11^{k+2} + 12^{2k+1}$ is divisible by 133.
Induction Step: Let $n = k + 1, k \geq 0$.

$$\begin{aligned} 11^{(k+1)+2} + 12^{2(k+1)+1} &= 11^{(k+2)+1} + 12^{2k+1+2} \\ &= 11^{(k+2)+1} + 12^{2k+1} \cdot 12^2 \\ &= 11^{(k+2)+1} + 12^{2k+1} \cdot (133 + 11) \\ &= 11 \cdot (11^{(k+2)} + 12^{2k+1}) + 133 \cdot 12^{2k+1} \end{aligned}$$

$(11^{(k+2)} + 12^{2k+1})$ is divisible by 133 by the hypothesis. It follows that, $11 \cdot (11^{(k+2)} + 12^{2k+1}) + 133 \cdot 12^{2k+1}$ is divisible by 133. Thus, $11^{(k+1)+2} + 12^{2(k+1)+1}$ is divisible by 133.

Hence, $11^{n+2} + 12^{2n+1}$ is divisible by 133 for all $n \geq 0$.

12. Let $\Sigma = \{a, b, c\}$ be the alphabet. Show that the number of words of length n in which the letter 'a' appears an even number of times is $(3^n + 1)/2$. Use induction or any other technique.
Solution:
$n = 1$. There are two strings, namely, b and c with even number of a's (no a's) and hence the base case is true. $(3^1 + 1)/2 = 2$.
Assume for a length n string in which a appears even number of times, it is $(3^n + 1)/2$.
Consider a $(n + 1)$ length string in which a appears even number of times. This can be obtained from n length string in two ways.
Let a_n denote the number of n-length strings containing even number of a's. a_{n+1} can be obtained from (i) a_n by appending the character b or c as the $(n + 1)^{th}$ character. (ii) to each n-length string containing odd number of a's append the character a as the $(n + 1)^{th}$ character. Note that there are 3^n length n strings, of which, $3^n - a_n$ is the number of strings with odd number of a's. Therefore, $a_{n+1} = a_n + a_n + (3^n - a_n)$, which is $(3 \cdot 3^n + 1)/2$. Hence, the claim.

13. Prove or Disprove. $(7 \times 7 - 1)$ chess board can be tiled using trio-minoes.
Solution: The claim is true. We shall obtain 16 different configurations of $(7 \times 7 - 1)$ and in each, we can argue that the tiling is possible. All other configurations are symmetric to one of 16 configurations. To tile $(7 \times 7 - 1)$, we visualize $(7 \times 7 - 1)$ as a combination of one or more of sub chess board of size $(4 \times 4 - 1)$, 2×3, 4×3. Further, in each of sub chess board, tiling is possible.

14. Recall currency change problem. Are R_3 and R_6 sound enough to give change for R_x. If not, construct an infinite set of counter examples to justify your claim.
Solution:
$A = \{x | x = 3k + 1 \text{ or } 3k + 2, k \text{ is an integer }\}$. If x is of this form, then change for R_x cannot be given using R_3 and R_6.

15. Generalization of currency change problem: Given the denominations; R_m and R_n, what is the condition for m and n such that change for R_x can be given. If the solution exists, what is the condition for m and n, what is the lower bound for the base case, and complete the argument using MI.
Solution:
`Approach:` 1 Frobenius number : Given m and n which are relatively prime, the largest number that

cannot be expressed as a linear combination of n and m is $mn-m-n$. This implies that starting from $mn-m-n+1$ onwards, we can express the number as $mx+ny$ for some positive integers x and y

`Approach: 2` (This approach was discovered by Aneesh team, Coe16) `Basis:` There are m base cases. $\{mn, mn+1, mn+2, \ldots, mn+(m-1)\}$ are the basis. Clearly, mn can be expressed as a linear combination using m and n. Assuming $m > n$, and m and n are relatively prime, we now show that each expression in basis set can be expressed as $mx - ny$ for some integers x and y. Towards this end, we define the following function.

$F(x, y) = (mx - ny) \bmod m$

It follows that, $F(x, x) = (mx - nx) \bmod m = x(m-n) \bmod m$

`Claim:` $F(1,1), F(2,2), \ldots, F(m-1, m-1)$ are distinct. The remainders given by the above function are distinct. We shall prove this by contradiction. Suppose, there exists k, l such that $1 \leq l < k \leq (m-1)$ and $F(k,k) = F(l,l)$.

$(m-n)k \bmod m = (m-n)l \bmod m$

$(m-n)(k-l) \bmod m$. This implies that m properly divides $(m-n)(k-l)$.

It follows that either m divides $(m-n)$ or m divides $(k-l)$.

Suppose, m divides $(m-n)$, then $m-n = mq$. Thus, we get, $n = m(1-q)$ and this implies that $q < 1$ which is not possible by the definition of positive quotient. If quotient can be negative, then n is a multiple of m which contradicts the fact that n and m are co-prime. Therefore, it may be the case that m divides $(k-l)$.

When m divides $(k-l)$, either $k = l$ or $k - l = mq$. This implies that, $k = l + mq$. Since, $q \geq 1$, k and l differ by m which is a contrdiction to the fact they both can differ by at most $(m-1)$. From the above arguments, it follows that, $k = l$. We again get a contradiction as indices are distinct. Thus, our assumption that $F(k,k) = F(l,l)$ is wrong and hence F values are distinct. This shows that starting from mn till $mn + (m-1)$, one can give change using m and n as per the linear combination defined by the function F. That is, $mn+1$ is given by $mn + x(m-n) \bmod m$ for some x. This completes the basis. For example, for denominations 3 and 5, we consider 3×5, $3\times5+1(5-3) \bmod 5, 3\times5+2(5-3) \bmod 5, 3\times5+3(5-3) \bmod 5, 3\times5+4(5-3) \bmod 5$. We see that $3\times5 = 3\times5+0(5-3) \bmod 5, 3\times5+1 = 3\times5+3(5-3) \bmod 5, 3\times5+2 = 3\times5+1(5-3) \bmod 5, 3\times5+3 = 3\times5+4(5-3) \bmod 5, 3\times5+4 = 3\times5+2(5-3) \bmod 5$. This shows that all integers in the base set can be expressed as a linear combination of 5 and 3.

Consider $R_x, x \geq mn + m$. Note that x can be written as $x = mn + rm + k \bmod m$, where r is the largest integer. The change for $mn + k \bmod m$ can be given as described in the base case and rm can be accounted by giving r R_m. Thus, for any R_x, the currency change using R_m and R_n is possible as established by the above arguments.

16. A monkey is expected to climb up a ladder of n steps. The Monkey can take either 1 step or 2 steps or 3 steps during each climbing. Thus, it is natural to get many different ways of climibing up a ladder. Present a good lower bound and an upper bound for the number of ways of climibing up the ladder on n steps. Prove your answer using MI.

    Solution:

    Let $T(n)$ denote the number of ways to climb up a ladder of n steps. Then,

    $T(n) = T(n-1) + T(n-2) + T(n-3), T(1) = 1, T(2) = 2, T(3) = (4)$.

    Good lower bounds: $T(n) \geq F_n$ which is (1.618^n) or $T(n) \geq (\frac{12}{7})^n$ or $T(n) \geq 3^{\frac{n}{3}}$.

    Good upper bounds: $T(n) \leq 2^n$ or $T(n) \leq 3^{n-1}$. The above bounds can be proved using MI.

17. Let $A = \{1, 2\}$. Construct the set $\rho(A) \times A$, where $\rho(A)$ is the power set (set of all subsets) of A.

    Solution:

    $A = \{1, 2\}$; $\rho(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

$\rho(A) \times A = \{(\phi, 1), (\phi, 2), (\{1\}, 1), (\{1\}, 2), (\{2\}, 1), (\{2\}, 2), (\{1, 2\}, 1), (\{1, 2\}, 2)\}$

18. Given that $A \subseteq C$ and $B \subseteq D$, show that $A \times B \subseteq C \times D$.
    Solution:
    To show that $A \times B \subseteq C \times D$, consider any arbitrary pair $(a, b) \in A \times B$, where $a \in A, b \in B$.
    $A \subseteq C \Rightarrow a \in C$ *and* $B \subseteq D \Rightarrow b \in D$. Thus, $(a, b) \in C \times D$.
    It follows that $A \times B \subseteq C \times D$.

19. Given that $A \times B \subseteq C \times D$, does it necessarily follow that $A \subseteq C$ and $B \subseteq D$?
    Solution:
    It is not necessary that if $A \times B \subseteq C \times D$ then, $A \subseteq C$ and $B \subseteq D$.
    Counter example:
    Let $A = \{1, 2\}, B = \phi$, $C = \{3\}$ and $D = \{4\}$
    $A \times B = \phi, C \times D = \{(3, 4)\}$
    Clearly, $A \times B \subseteq C \times D$ but $A \nsubseteq C$

20. Is it possible that $A \subseteq A \times A$ for some set A ?
    Solution:
    Yes. If $A = \phi$ then $A \subseteq A \times A$.

21. For each of the following check whether 'R' is Reflexive, Symmetric, Anti-symmetric, Transitive, an equivalence relation, a partial order.

    (a) $R = \{(a, b) \mid a - b$ is an odd positive integer $\}$.

    (b) $R = \{(a, b) \mid a = b^2$ where $a, b \in I^+\}$.

    (c) Let P be the set of all people. Let R be a binary relation on P such that (a, b) is in R if a is a brother of b.

    (d) Let R be a binary relation on the set of all strings of $0's$ and $1's$, such that $R = \{(a, b) \mid a$ and b are strings that have same number of $0's\}$.

    Solution:

Q.No	Reflexive	Symmetric	Anti-symmetric	Transitive	Equivalence	Poset
1.	$\times$	$\times$	✓	$\times$	$\times$	$\times$
2.	$\times$	$\times$	✓	$\times$	$\times$	$\times$
3.	$\times$	$\times$	$\times$	✓	$\times$	$\times$
4.	✓	✓	$\times$	✓	✓	$\times$

22. Let R be a symmetric and transitive relation on set A. Show that if for every 'a' in A there exists 'b' in A, such that (a, b) is in R, then R is an equivalence relation.
Solution:
Given: $\forall a \, \exists b \, (b \in A \wedge (a, b) \in R)$.
To prove: R is reflexive

Since R is symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R$ and since R is transitive, $(a, b) \in R, (b, a) \in R \Rightarrow (a, a) \in R$ and this argument is true $\forall a \in A$. Therefore, R is reflexive. Hence R is an equivalence relation.

23. Let R be a transitive and reflexive relation on A. Let T be a relation on A, such that (a, b) is in T if and only if both (a, b) and (b, a) are in R. Show that T is an equivalence relation.
Solution:
To prove that T is equivalence relation we need to prove T is reflexive, T is symmetric and T is transitive.
Given that $(a, b) \in T$ iff $(a, b), (b, a) \in R$
Clearly $(a, a) \in T \; \forall a \in A$, This is true because R is reflexive. This proves that T is reflexive.
If $(a, b) \in T$ we need to prove that $(b, a) \in T$. By the hypothesis (given condition), it is easy to see that $(b, a) \in T$. Hence T is symmetric.
If $(a, b) \in T$ and $(b, c) \in T$, we need to prove that $(a, c) \in T$.
$(a, b) \in T \rightarrow (a, b), (b, a) \in R$
$(b, c) \in T \rightarrow (b, c), (c, b) \in R$
Since R is transitive $(a, c) \in R$ and $(c, a) \in R$, this implies that $(a, c) \in T$. Hence T is transitive. Therefore, T is an equivalence relation.

24. Let R be a binary relation. Let $S = \{(a, b) \mid (a, c) \in R \text{ and } (c, b) \in R \text{ for some } c\}$. Show that if R is an equivalence relation, then S is also an equivalence relation.
Solution:
To Prove: S is reflexive.
Since R is reflexive $(a, a) \in R \; \forall a \in A$. Clearly $(a, a) \in S \; \forall a \in A$. This proves that S is reflexive.
To prove:S is symmetric
$(a, b) \in S \rightarrow \exists x \; (a, x) \in R, (x, b) \in R$
Since R is symmetric $(x, a) \in R, (b, x) \in R$.
Therefore by given definition, $(b, a) \in S$.
This proves that S is symmetric.
To prove: S is transitive
If $(a, b) \in S$ and $(b, c) \in S$ we need to prove that $(a, c) \in S$.
$(a, b) \in S \rightarrow \exists d \; (a, d), (d, b) \in R$
R is symmetric $\rightarrow (d, a), (b, d) \in R$
$\Rightarrow (a, b) \in R, (b, a) \in R$
$(b, c) \in S \rightarrow \; \exists e \; (b, e), (e, c) \in R$
R is symmetric $\Rightarrow (e, b), (c, e) \in R$
$\Rightarrow (b, c) \in R, (c, b) \in R$
Since R is transitive, $(a, c) \in R, (c, a) \in R$ ——(1)
Since R is reflexive, $(c, c) \in R$ ——(2)

From (1) and (2) it follows that $(a, c) \in S$

Therefore, S is transitive and hence an equivalence relation.

25. Let R be a reflexive relation on a set A. Show that R is an equivalence relation if and only if (a, b) and (a, c) are in R implies that (b, c) is in R.

Solution:

Necessity: Given that R is an equivalence relation, we need to prove that $(a,b),(a,c) \in R \rightarrow (b,c) \in R$
Since R is symmetric, $(a,b) \in R \Rightarrow (b,a) \in R$
Since R is transitive, $(b,a),(a,c) \in R \Rightarrow (b,c) \in R$
Hence necessity is proved.

Sufficiency: To show that R is an equivalence relation, we need to show that R is symmetric and transitive.
By definition, $(a,b),(a,c) \in R \Rightarrow (b,c) \in R$
Also $(a,c),(a,b) \in R \Rightarrow (c,b) \in R$
Therefore, R is symmetric.
To prove transitivity, if $(x,y),(y,z) \in R$ then $(x,z) \in R$
$(x,y) \in R,\ (a,x)\&(a,y) \in R$
$(y,z) \in R,\ (a,y)\&(a,z) \in R$
$(a,x)\&(a,z) \in R \Rightarrow (x,z) \in R$. Hence R is transitive.
Therefore R is an equivalence relation. Hence sufficiency is proved.

26. Let A be a set with n elements. Using mathematical induction,

    (a) Prove that there are 2^n unary relations on A.

    (b) Prove that there are 2^{n^2} binary relations on A.

    (c) How many ternary relations are there on A ?

    Solution:

    1. Let us prove this by induction on number of elements in A, n.
       Base Case: If $n = 0$ then, number of relations is $2^0 = 1$ (Empty set). If $n = 1$ then, number of unary relations is $2 = 2^1$ (If $A = \{x\}$ then, unary relations on $A = \{\phi, x\}$)
       Hypothesis: Assume that the statement is true for $n = k, k \geq 1$
       Induction Step: Let A be the set with $n = k + 1$ elements, $k \geq 1$.
       Number of unary relations on a set with $k + 1$ elements = Number of unary relations on a set with k elements + 2^k ($(k+1)^{th}$ element can be placed in each of 2^k subsets of k elements) = 2^k + 2^k = 2^{k+1}.
    2. Let us prove this by induction on number of elements in A, n.
       Base Case: If $n = 0$ then, number of relations is $2^0 = 1$ (Empty set). If $n = 1$ then, number of binary relations is $2 = 2^{1^2}$ (If $A = \{x\}$ then, $A \times A = \{\phi, (x,x)\}$).
       Hypothesis: Assume that the statement is true for $n = k, k \geq 1$
       Induction Step: Let A be the set with $n = k+1$ elements, $k \geq 1$. Let $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$
       For k elements, number of binary relations are 2^{k^2}. For $(k+1)^{th}$ element, we have the following $2k + 1$ binary elements:
       (x_1, x_{k+1}), (x_2, x_{k+1}), …, (x_k, x_{k+1}), (x_{k+1}, x_1), (x_{k+1}, x_2), …, (x_{k+1}, x_k), (x_{k+1}, x_{k+1}).
       Therefore, number of binary relations for the set $A = 2^{k^2}\ .\ 2^{2k+1} = 2^{k^2+2k+1} = 2^{(k+1)^2}$.

    3. Number of ternary relations on $A = 2^{n^3}$

27. If P is a prime greater than 3, then P^2 has the form $12k + 1$, where k is an integer.

Solution 1:
Since P is a prime number, P is an odd number. This implies, P^2 is an odd number. Thus, $P^2 - 1$ (even number) is divisible by 4 ($P^2 - 1$ can be written as $(P+1)(P-1)$, where $P+1$ is even and $P-1$ is even). Also, $P^2 - 1$ is divisible by 3 (for any three consecutive integers $P-1, P, P+1$, any one integer is divisible by 3. But, P is a prime number and > 3. Therefore, either $P-1$ or $P+1$ is divisible by 3). We know that, Let $a \mid b$ and $c \mid b$. If $gcd(a, c) = 1$ then, $ac \mid b$. Since $gcd(4, 3) = 1$, 12 divides $P^2 - 1$ i.e., $P^2 - 1 = 12k$, where k is any integer. Thus, $P^2 = 12k + 1$.

Solution 2:
Any prime number > 3 can be written in the form $6m \pm 1$, where m is a positive integer. Thus, $P^2 = 36m^2 \pm 12m + 1 = 12(3m^2 \pm m) + 1 = 12k + 1$, where $k = 3m^2 \pm m$.

28. If an integer is simultaneously a square and a cube (ex: $64 = 8^2 = 4^3$), verify that the integer must be of the form $7n$ or $7n + 1$.

Solution:
Direct proof: Let $z = x^2$ and $z = y^3$ for some $x, y \in \mathbb{I}$. Note that any number $x \in \mathbb{I}$ can be represented as $x \ mod\ 7 = i$, $0 \leq i \leq 6$. This implies $x^2 \ mod\ 7 = j$, $j = \{0, 1, 2, 4\}$. Similarly, $y \ mod\ 7 = i$, $0 \leq i \leq 6$ implies that $y^3 \ mod\ 7 = k$, $k = \{0, 1, 6\}$. It follows that if $z \ mod\ 7 = 0$ or $z \ mod\ 7 = 1$. Therefore, $z = 7n$ or $z = 7n + 1$.

29. The circumference of a 'roulette wheel' is divided into 36 sectors to which the numbers $1, 2, \ldots, 36$ are assigned in some arbitrary manner. Show that there are 3 consecutive sectors such that the sum of their assigned numbers is at least 56.

Solution:
Let a_i denotes the sum of three consecutive sectors from sector i, $1 \leq i \leq 36$. On the contrary, assume that the sum of any three consecutive sectors is ≤ 55. Therefore,

$$\begin{aligned}
\sum_{i=1}^{36} a_i &\leq 36 \times 55 \\
3 \times (1 + 2 + \ldots + 36) &\leq (36 \times 55) \\
3 \times \frac{36 \times 37}{2} &\leq (36 \times 55) \\
111 &\leq 110 \quad (\textit{ a contradiction})
\end{aligned}$$

Thus, our assumption that the sum of any three consecutive sectors is ≤ 55 is wrong. Therefore, there exist 3 consecutive sectors such that the sum of their assigned numbers is at least 56.

30. If there are 104 different pairs of people who know each other at a party of 30 people, then show that some person has 6 or fewer acquaintances.

Solution:
On the contrary assume that all persons are having at least 7 acquaintances. Therefore, the number of distinct acquaintance pair is at least $30 \times 7/2 = 105$. This is contradiction to the fact that there are 104 different pair of acquaintances. Therefore, our assumption is wrong and it follows that there exist at least a person with 6 or fewer acquaintances.

31. Prove by induction : For $n \geq 1$, $8^n - 3^n$ is divisible by 5.

Solution:
Base case: $n = 1$, $8^1 - 3^1$ is divisible by 5. *Induction hypothesis:* Assume that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$. *Induction step:* For $n \geq 1$, consider $8^{n+1} - 3^{n+1} = (8.8^n - 3.3^n) = (5+3)8^n - 3.3^n$ $= 5.8^n + 3(8^n - 3^n)$. By the induction hypothesis, $8^n - 3^n$ is divisible by 5 and hence, $= 5.8^n + 3(8^n - 3^n)$ is divisible by 5. Therefore, we can conclude that $8^n - 3^n$ is divisible by 5 for all $n \geq 1$.

32. Prove by induction: a number, given its decimal representation is divisible by 3 iff the sum of its digits is divisible by three.

    Solution:
    Let us prove this by induction on number of digits, n.
    Base Case: $n = 1$, clearly, then a number is divisible by 3 iff the sum of the digits is divisible by 3. For example, the numbers 3,6,9 satisfy this case.
    Hypothesis: Assume that the statement holds for $n = k$, $k \geq 1$. i.e., a number composed of k digits is divisible by 3 iff the sum of its digits is divisible by 3.
    Induction Step: Let $n = k+1$, $k \geq 1$. Let x be a number composed of $k+1$ digits. Our claim is to prove that x is divisible by 3 iff the sum of the digits in x is divisible by 3.

    Let the decimal expansion of $x = a_k a_{k-1} \dots a_0$ be,
    $x = a_k 10^k + a_{k-1} 10^{k-1} + \dots + a_0$. $\qquad ----------> (1)$

    Since, $a_m 10^m$ can be written as $a_m + a_m(10^m - 1)$, equation (1) can be written as follows:

    $x = (a_k + a_{k-1} + \dots + a_0) + (a_k(10^k - 1) + a_{k-1}(10^{k-1} - 1) + \dots + a_1(10^1 - 1))$,

    Since, 3 divides $10^n - 1$, implies that 3 divides $(a_k(10^k - 1) + a_{k-1}(10^{k-1} - 1) + \dots + a_1(10^1 - 1))$.
    By the hypothesis, $(a_{k-1} + \dots + a_0)$ is divisible by 3 iff $a_{k-1} \dots a_0$. Thus, x is divisible by 3 iff a_k is divisible by 3. i.e., x is divisible by 3 iff $(a_k + a_{k-1} + \dots + a_0)$ is divisible by 3. Therfore, x is divisible by 3 iff the sum of the digits in x is divisible by 3.

33. Show that any integer composed of 3^n identical digits is divisible by 3^n. (for example: 222 is div by 3, 555,555,555 is div by 9)
    Solution:
    We shall prove this by induction on n.
    Base Case: For $n = 1$, we note that any 3-digit integer with 3 identical digits is divisible by 3. Since, for any $k \in \{1, \dots, 9\}$, $kkk = k \cdot (111)$. Further, 111 is divisible by 3. Therefore, kkk is divisible by 3.
    Hypothesis: Assume that the statement is true for $n = k$, $k \geq 1$.
    Induction Step: For $n = k+1, k \geq 1$. Let x be an integer composed of 3^{k+1} identical digits. We note that x can be written as
    $x = y \times z$
    where y is an integer composed of 3^k identical digits, and $z = 10^{2 \cdot 3^k} + 10^{3^k} + 1$.
    For example, $x = 666666666 = 666 \times 1001001 = y \times (10^{2 \cdot 3^1} + 10^{3^1} + 1)$. y is divisible by 3^k by the hypothesis and z is divisible by 3 (sum of the digits is divisible by 3). Thus x is divisible by 3^{k+1}.

34. A person takes at least one tablet a day for 50 days. He takes 90 tablets altogether. Is it true that during some sequence of consecutive days he has taken exactly 24 tablets. Justify your answer.
    Solution:
    Let a_i be the number of tablets the patient has taken till the end of the i^{th} day. Thus we have the following sequence:
    $1 \leq a_1 < a_2 < \dots < a_{50} = 90$.

Thus we have
$1 + 24 \leq a_1 + 24 < a_2 + 24 < \ldots < a_{50} + 24 = 90 + 24$.
i.e.,
$25 \leq a_1 + 24 \leq a_2 + 24 \leq \ldots \leq a_{50} + 24 = 114$.
Thus, among all the numbers: $a_1, a_2, \ldots, a_{50}, a_1 + 24, \ldots, a_{50} + 24$; there are 100 numbers (pigeons) and 114 (pigeon holes). So, there is no possibility of two numbers to be equal (Since a patient takes at least one tablet a day). Thus, there is no sequence of consecutive days where the patient has taken exactly 24 tablets.

35. Show that one of any n-consecutive integers is divisible by n.
Solution:
On the contrary, we assume that there does not exist a number divisible by n in a set of n consecutive integers. We can place integer i in congruence class j, where $j = i \ mod \ n$, $1 \leq j \leq n-1$ corresponding to pigeon holes. Observe that n integers (pigeons) are there and by pigeonhole principle, there exist a class with more than one integer, say a, b where $a = x.n + r$ and $b = y.n + r$. Note that x and y differ by at least one and it follows that there exist at least $n + 1$ consecutive integers from a to b inclusive of both. This is a contradiction to the fact that there are n consecutive integers. Therefore our assumption is wrong and one of any n-consecutive integers is divisible by n.

36. Show that among $(n + 1)$ positive integers less than or equal to $2n$, there are 2 consecutive integers.
Solution:
Pigeon holes: $(1, 2), (3, 4), \ldots, (2n - 1, 2n)$, n pigeon holes
Pigeons: $n + 1$ pigeons. Choose $n + 1$ distinct numbers from the $2n$ positive integers. Place pigeon x in the hole (a, b) if $a = x$ or $b = x$.
PHP: At least 2 pigeons will be placed in a hole and since $(n+1)$ integers are distinct those two pigeons are consecutive numbers (by the definition of pigeon holes).

37. Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.
Solution:
Consider a person A and divide the remaining 4 persons into two sets, friends and the enemies of A. There exist at least two persons the friends set or in the enemies set of A by pigeonhole principle. We can see the below possibilities

Cardinality of friend set of A	**Cardinality of enemy set of A**
2	2
1	3
3	1
0	4
4	0

Consider the possibility where there exist 2 friends B, E and 2 enemies C, D of A. If B and E are friends, then there exist three mutual friends, $\{A, B, E\}$. Therefore, we consider a scenario where B and E are enemies. Similarly, if C and D are enemies, then there exist three mutual enemies, $\{A, C, D\}$. Therefore, we consider a scenario where C and D are friends. Now if B is a friend of C, and D is a friend of E, then there does not exist three mutual friends or three mutual enemies in the scenario. Scenario in short: friend relations $(A, B), (B, C), (C, D), (D, E), (E, A)$.

38. Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.

Solution:
We shall prove there are either three mutual friends or four mutual enemies and the argument for the other claim is symmetric.
Let $A, B, \ldots, J$ be the ten persons. Take a person A: divide the remaining 9 persons into friends set of A and the enemies set of A. By PHP, $\lceil \frac{9}{2} \rceil = 5$, at least five persons either in the friends set of A or in the enemies set of A. Therefore, the possibilities are

We present case by case analysis among the above possibilities: at least 4 friends for A or at least 6

Cardinality of friend set of A	**Cardinality of enemy set of A**
5	4
6	3
7	2
8	1
9	0
4	5
3	6
2	7
1	8
0	9

enemies for A.
Case 1: At least 4 friends for A
With respect to A, assume that (B, C, D, E) are the 4 friends. If any two of (B, C, D, E) are friends, then those two along with A forms 3 mutual friends. If none of them are friends, then (B, C, D, E) form 4 mutual enemies.
Case 2: At least 6 enemies for A
w.l.o.g. assume that (B, C, D, E, F, G) are the 6 enemies for A. W.k.t. Among 6 people, there exist either 3 mutual friends or 3 mutual enemies. If there are 3 mutual friends then there is nothing to prove or if there are three mutual enemies then this 3 along with A form four mutual enemies.

39. Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$ then there are always two which differ by 1.
Solution:
Consider the groups $\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$ as pigeon holes. The $n+1$ distinct integers form the pigeons and by pigeon hole principle, there exist a group g_k containing more than one integer; say $i, i+1 \in g_k$. This implies that among the $n+1$ distinct integers, there exist two $\{i, i+1\}$ which differ by 1.

40. Given 8 distinct integers $(x_1, x_2, \ldots, x_8)$, show that there exist a pair with the same remainder when divided by 7.
Label the pigeonholes with possible remainders when a number is divided by 7. i.e., labels are $0, 1, \ldots, 6$. Thus there are 7 pigeonholes, and given that 8 distinct integers, by pigeonhole principle, there exist at least one label (pigeon hole) having more than one integer. This implies that there exist a pair of integers with same remainder when divided by 7.

41. Given 7 distinct integers, there must exist two integers such that the sum or difference is divisible by 6.

Like before, pigeon holes denote possible remainders when a number is divided by 6. Due to 7 distinct integers, by pigeonhole principle, there exist at least one hole having more than one integer, say a, b. It follows that difference of a and b is a multiple of 6. If both a and b leave remainder '3' then sum or difference is divisible by 6.

42. Given $n+1$ distinct integers, then there is some pair of them such that their difference is divisible by the positive integer n.
    Each pigeonhole groups integers having same remainder when divided by n. Thus there are n pigeonholes and $n+1$ distinct integers. By pigeonhole principle, there exist at least one remainder class having more than one integer, say a, b. It follows that $a = n.x + r$ and $b = n.y + r$. Without loss of generality, let $a > b$. This implies that $x > y$ and $a - b = n(x - y)$. Therefore, difference of a and b is a multiple of n.

43. Given 37 distinct positive integers, then there must be at least 4 of them that have the same remainder when divided by 12.
    Consider the pigeonholes to be the class of integers having same remainder when divided by 12. There exist $37 = 12 \times 3 + 1$ distinct positive integers (*pigeons*) and 12 remainder classes (*pigeonholes*). Therefore by generalized pigeonhole principle, there exist a remainder class having at least $3+1 = 4$ integers. Therefore there exist at least 4 distinct integers with same remainder.

44. A student has 37 days to prepare for an examination. From past experience she knows that she will require no more than 60 hours of study. She also knows that she wishes to study at least 1 hour per day. Show that no matter how she schedules her study time (a whole number of hours per day however) there is a succession of days during which she would have studied exactly 13 hours.
    Let s_i, $1 \leq i \leq 37$ be the number of hours studied till i^{th} day. Then,
    $s_1 < s_2 < \ldots < s_{37} \leq 60$.
    $s_1 + 13 < s_2 + 13 < \ldots < s_{37} + 13 \leq 60 + 13 = 73$.
    Note that for all $1 \leq i, j \leq 37$, $s_i \neq s_j$ where $i \neq j$. There exist $2 \times 37 = 74$ summands (*pigeons*), and 73 distinct integer values (*pigeonholes*). Therefore, there exist two summands having same value. i.e., $s_i = s_j + 13$ and this implies that $s_i - s_j = 13$. Thus there exist a period of $i - j$ consecutive days (day $j+1, \ldots, i$) in which she spent 13 hours for studying.

45. Given n pigeons to be distributed among k pigeonholes:
    What is a necessary and sufficient condition on n and k that, in every distribution, at least two pigeonholes must contain the same number of pigeons.
    Consider the scenario in which pigeonhole P_i is filled with i pigeons, $0 \leq i \leq k-1$. If P_k is filled with any value between 0 and $k-1$, then there are two pigeon holes containing the same number of pigeons. Moreover, if P_k contains any number greater than $k-1$ (k or more) then our claim need not be true always. Therefore, the number of pigeons $n = x + y$, where $x = (0 + 1 + \ldots + k - 1)$ and $y \leq k - 1$. i.e. $n \leq \frac{(k-1)k}{2} + k - 1$.

46. What is the value of n (minimum n) such that in any group of n people you see either 3 mutual enemies or 4 mutual friends.

Minimum value of n for which in any group of n people there exist either 3 mutual enemies or 4 mutual friends is 9. Consider 9 persons $p_1, p_2, \ldots, p_9$ and note that

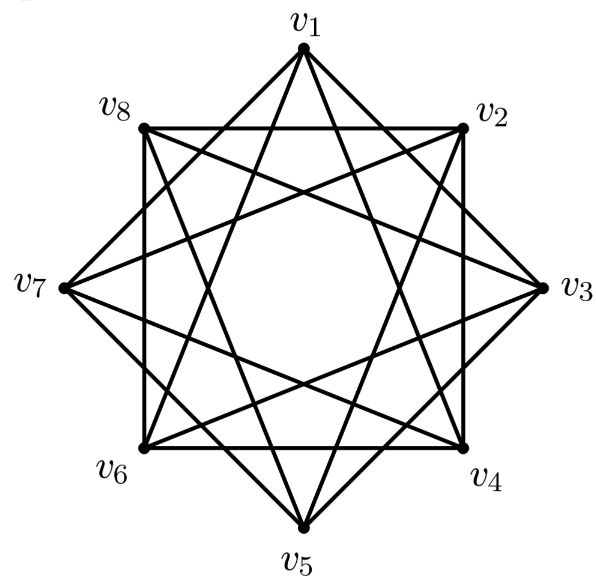

Figure 1: Fig: An illustration on $n = 8$

In the above figure, node v_i, $1 \leq i \leq 8$ represents people and edge between node represent friend relation between them. The figure is a counter example illustrating that in a group of 8 people, there need not be 3 mutual enemies or 4 mutual friends. Now we prove that there exist either 3 mutual enemies or 4 mutual friends among a group of 9 people. Consider 9 nodes representing people with every pair of vertices connected by an edge representing relationships. The edge is colored *Blue* for friend and *Red* for enemy relationships. We now show that their exist a red triangle (representing 3 mutual enemies) or blue K_4 (4 vertices with 6 edges among them) representing 4 mutual friends. Note $N_B(x) = \{y \mid (x, y) \textit{ is blue}\}$ and $N_R(x) = \{y \mid (x, y) \textit{ is red}\}$

Case 1: If there exist a vertex v_i with at least 6 blue edges incident on it, then in $N_B(v_i)$, there exist either a red triangle induced on $R \subseteq N_B(v_i)$ or blue triangle B' induced on $B \subseteq N_B(v_i)$ as $|N_B(v_i)| = 6$. Therefore there exist a red triangle induced on R or a blue K_4 induced on $B' \cup v_i$ in the graph.

Case 2: If there exist a vertex v_i with at least 4 red edges incident on it, then we can see the following. If there exist $v_j, v_k \in N_R(v_i)$, such that (v_j, v_k) is red, then $\{v_i, v_j, v_k\}$ induces a red triangle. On the other hand, if there does not exist $v_j, v_k \in N_R(v_i)$, such that (v_j, v_k) is red, then $\{v_i\} \cup N_R(v_i)$ has an induced K_4.

Case 3: All the vertices v_i in the graph are having at most 3 red edges and at most 5 blue edges incident on them. Since all edges are colored either blue or red, it follows that in the graph all vertices have exactly 3 red edges and 5 blue edges incident on them. It follows that there are $(9 \times 3)/2 = 13.5$ red edges, similarly, there are $(9 \times 5)/2 = 22.5$ blue edges. Note that sum of the degrees in a graph is twice the number of edges, which implies that the number of edges is always even. Further, the set of edges can be partitioned into 'red' and 'blue' edges. In our case, degree sums due to 'red' edges and 'blue' edges yield a non-integer, a contradiction. Therefore, this case does not occur. Among the mutually exclusive and exhaustive cases discussed, case $1, 2$ shows that there exist either a red triangle or a blue K_4 and in case 3 such a graph does not exist. Therefore there exist 3 mutual enemies (red triangle) or 4 mutual friends (blue K_4) in a group of at least 9 people (vertices).

47. Using PIE (principle of inclusion and exclusion), Find the number of positive integers not exceeding 100 that are either odd or the square of an integer
Solution:
Number of odd numbers = $|O| = 50$ (half of 100)
Number of square numbers = $|S| = 10$ (square root of 100)
Number of odd square numbers = $|O \cap S| = 5$ (half of the above number)
$|O \cup S| = |O| + |S| - |O \cap S|$
$= 50 + 10 - 5 = 55$