Domination and Hamiltonicity Variants in Subclasses of Split Graphs

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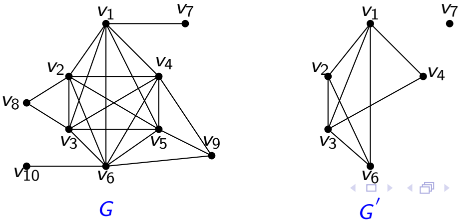
- A Graph $G'(V',E')$ is called a sub graph of $G(V,E)$ if $V'(G') \subseteq V(G)$ and $E'(G') \subseteq E(G)$.
- Induced sub graph - $\forall u,v \in V'(G'), \{u,v\} \in E'(G') \leftrightarrow \{u,v\} \in E(G)$.
- H-free graphs - no induced sub graph H
- Chordal graphs - every induced cycle $C_{n\geq 4}$ has a chord.
- Split graphs $G(K \cup I, E)$ - $2K_2$-free chordal graphs. K- clique, I- Independent set.
    - for every $v \in K$, $d_G^I(v) = |N_G(v) \cap I|$
    - $\Delta^I = \max\{d_G^I(v) : v \in K\}$



Graph Theoretic Preliminaries

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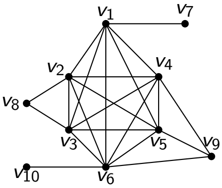

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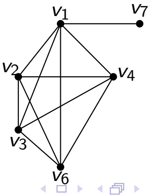

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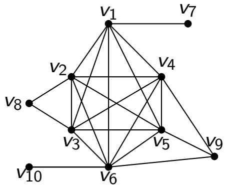

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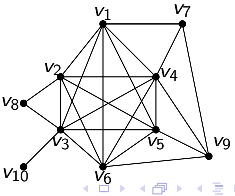

ChordalGraph

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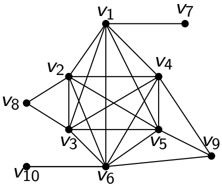

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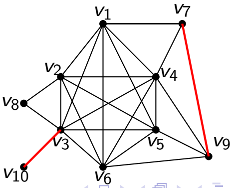

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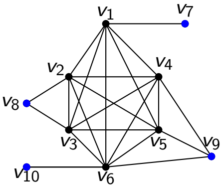
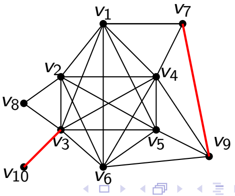
$\Delta^I = 2$



SplitGraph



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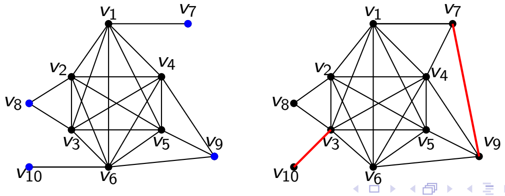

Our Contribution

- Polynomial-time algorithms for the following problems in $K_{1,3}$-free split graphs (Claw-free split graphs)

  - Minimum Total Outer-Connected Dominating set
  - Longest Path
  - Maximum-Cut
  - Steiner Path

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- ⋆ NP-Complete results for the following problems in $K_{1,5}$-free split graphs
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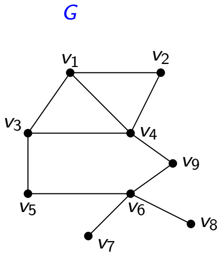
Definitions

- ⋆ A *dominating set* (DS) of graph $G(V, E)$ is a set of vertices $D \subseteq$ V such that every vertex in $V \setminus D$ is adjacent to a vertex in D.
- ⋆ A set $D \subseteq V$ is called a *total dominating set* (TDS) of graph $G(V, E)$ if $\forall v \in V$, $|N_G(v) \cap D| \geq 1$.
- ⋆ A total dominating set D of $G(V, E)$ is called a *total outer-connected dominating set* (TOCD) if $G[V \setminus D]$ is connected.



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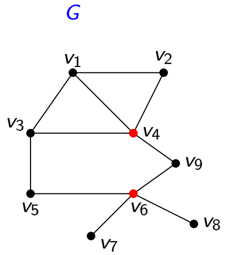

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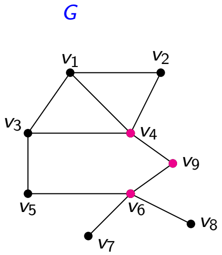

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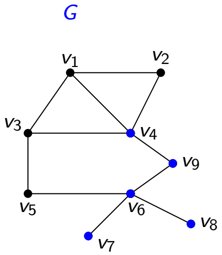

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Known Results on MTOCD

MTOCD - Minimum Total Outer-Connected Domination Problem

MTOCD (G)

Instance: Connected Graph $G(V, E)$
Question: Find a minimum set $D \subseteq V(G)$ such that every vertex in G is adjacent to a vertex in D and G[V\D] is connected?

Graph class	Complexity of MTOCD problem
Bipartite graphs	NPC [a]
Chordal graphs	NPC [b]
Split graphs	NPC [b]
Trees	P [b]

[a] O.Favaron, H.Karami, and S.M.Sheikholeslami: On the total outer-connected domination in graphs. *Combinatorial Optimization* (27), 451-461, (2014).

[b] B.S.Panda and A. Pandey: Complexity of total outer-connected domination problem in graphs. *Discrete Applied Mathematics* (199), 110-122, (2016).

Theorem

Let $G(K \cup I, E)$ be a connected claw-free split graph with $|I| \geq 4$ and P is the set of pendant vertices. Then G has a minimum total outer-connected dominating set D such that

$$|D| = \begin{cases} |I| + |P| - 1, & \text{if } |I| = |P| \text{ and } |N(I)| = |K|. \\ |I| + |P|, & \text{otherwise.} \end{cases} \quad (1)$$

Theorem

For a connected claw-free split graph $G(K \cup I, E)$ such that $|I| \leq 3$ and D is a minimum total outer-connected dominating set of G, then $2 \leq |D| \leq 6$.

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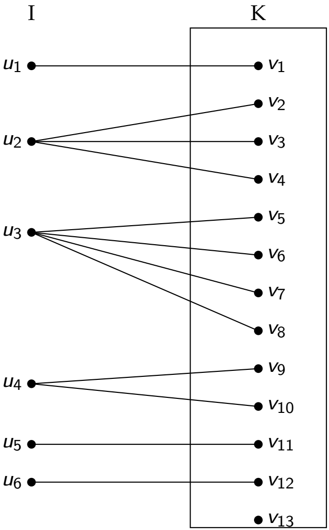
MTOCD: P-time Algorithm in claw-free split graphs

⋆ For a claw-free split graph G, if $|I| \geq 4$ then $\Delta_G^I \leq 1$.



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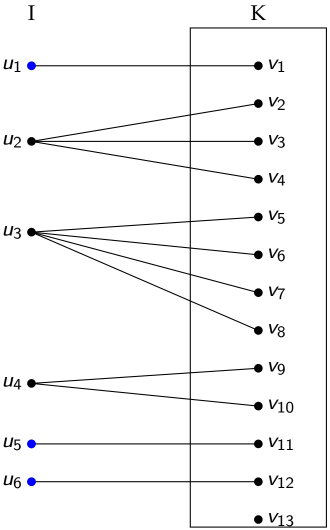

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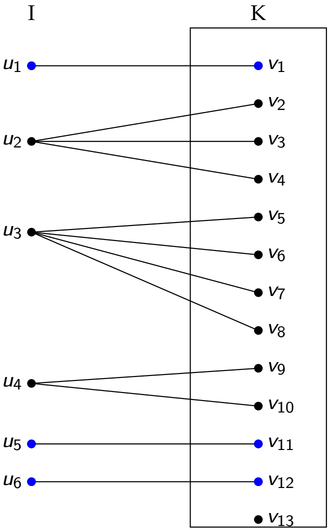

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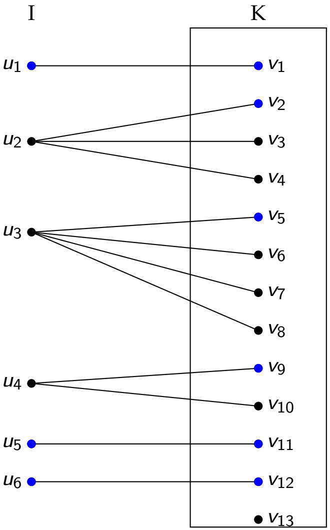

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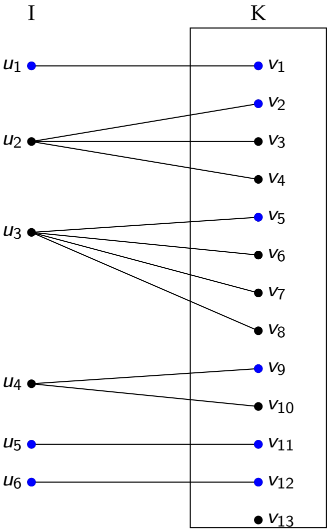

$$|D| = |I| - |P| + 2|P|$$

$$\Rightarrow |D| = |I| + |P|$$

Longest Path

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- ⋆ The Longest path problem is the problem of finding a simple path of maximum length in a given graph.
- ⋆ As the Longest path problem is a generalization of the Hamiltonian path problem, it is natural to see the tractability of Longest path problem on the classes of graphs where Hamiltonian path problem is polynomial-time solvable.
- ⋆ The Hamiltonian path problem is polynomial-time solvable in $K_{1,3}$-free split graphs and in $K_{1,4}$-free split graphs.[a]

[a] P.Renjith and N.Sadagopan: Hamiltonian path in $K_{1,t}$-free split graphs - A dichotomy. *CALDAM* (4),30-44,(2018).

Theorem

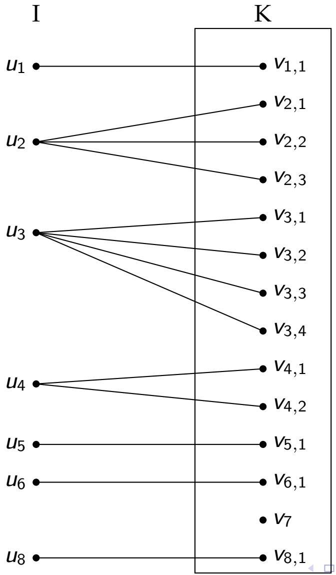
Let $G(K \cup I, E)$ be a connected claw-free split graph with $|I| \geq 4$ and P is the set of pendant vertices, the longest path Q in G has a length

$$\begin{cases} |K \cup I| - |P| + 2, & \text{if P} \geq 2. \\ |K \cup I|, & \text{otherwise.} \end{cases} \tag{2}$$

Theorem

Let $G(K \cup I, E)$ be a connected claw-free split graph with $|I| \leq 3$, then the longest path Q in G has a length $|K \cup I|$ or $|K \cup I| - 1$.

Longest Path: Algorithm in claw-free split graphs



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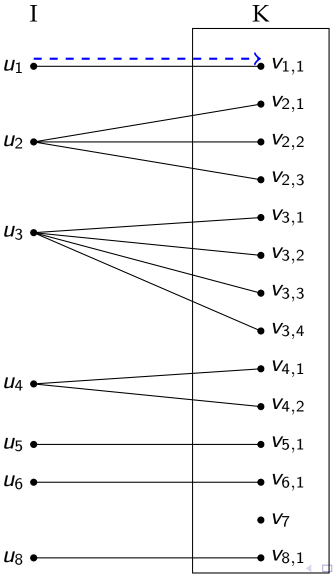

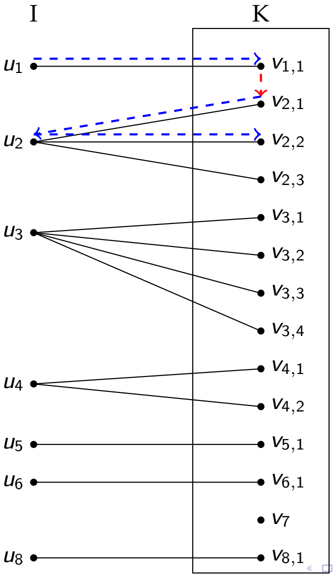
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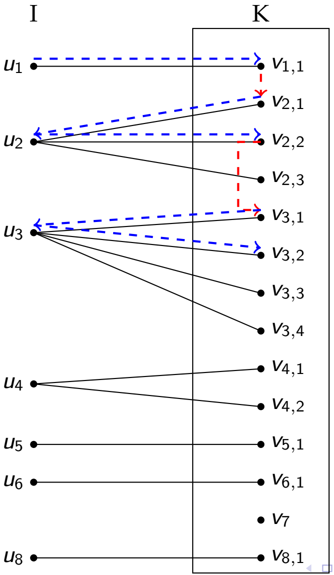

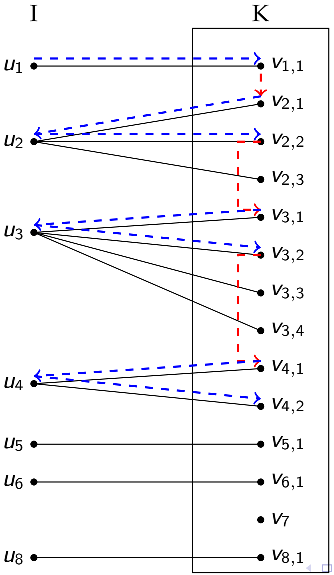
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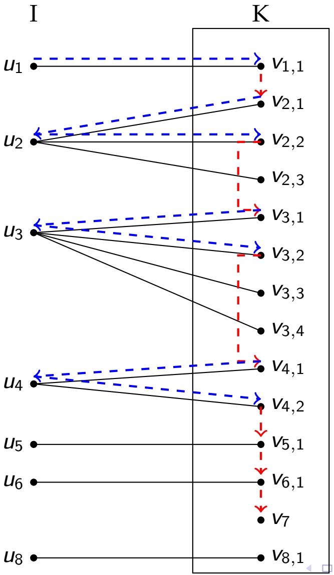

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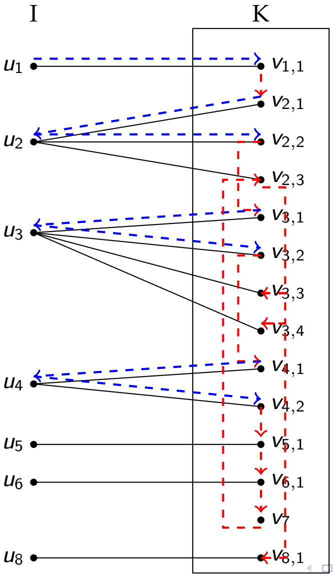

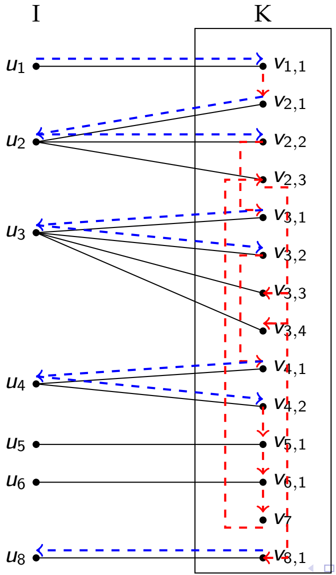
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Maximum-Cut

Maximum-Cut(G)

Instance: Graph $G(V, E)$
Question: Find a set $S \subseteq V(G)$ such that $|\delta(S)|$ is maximum, where $\delta(S) = \{\{u, v\} \mid u \in S \text{ and } v \in V \setminus S\}$?

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- ⋆ Maximum-Cut problem is NP-complete in 2-split graphs [a]. A split graph $G(K \cup I, E)$ is a 2-split graph if every vertex in I is adjacent to exactly two vertices in clique K.
- ⋆ We consider the Maximum-Cut problem in claw-free split graph $G(K \cup I, E)$ such that $\forall v \in I, d(v) \leq 2$.

[a] H.L.Bodlaender and K.Jansen: On the complexity of the maximum cut problem. *Nordic Journal of Computing* (7),1431,(2000)

Observation

The maximum-cut value of a clique K is $\lfloor \frac{|K|}{2} \rfloor \lceil \frac{|K|}{2} \rceil$.

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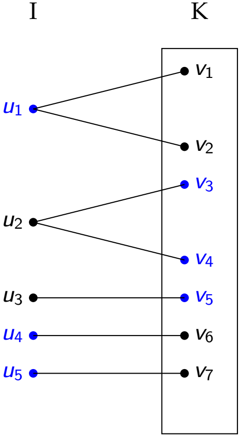
Lemma

For a connected claw-free split graph $G(K \cup I, E)$, such that $|I| \geq 4$, $\forall v \in I, d(v) = 1$ and $N_G(I) = K$, then maximum-cut value = $\lfloor \frac{|K|}{2} \rfloor \lceil \frac{|K|}{2} \rceil + |\delta(I)|$.

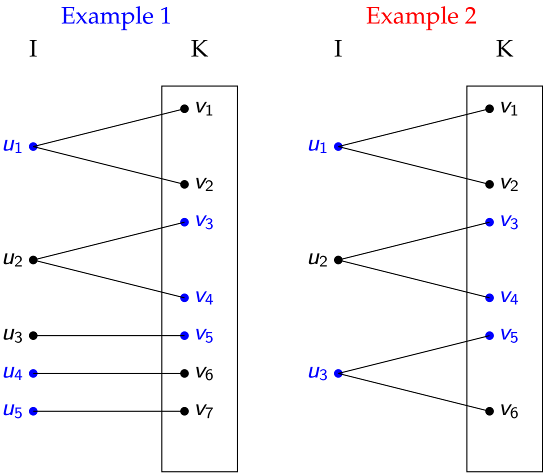
Theorem

For a connected claw-free split graph $G(K \cup I, E)$, such that $|I| \geq 4$, $\forall v \in I, d(v) \leq 2$, then maximum-cut value = $\lfloor \frac{|K|}{2} \rfloor \lceil \frac{|K|}{2} \rceil + |\delta(I)|$ or $\lfloor \frac{|K|}{2} \rfloor \lceil \frac{|K|}{2} \rceil + |\delta(I)| - 1$.

Example 1



$S = \{u_1, u_4, u_5, v_3, v_4, v_5\}$



$S = \{u_1, u_4, u_5, v_3, v_4, v_5\}$ $S = \{u_1, u_3, v_3, v_4, v_5\}$

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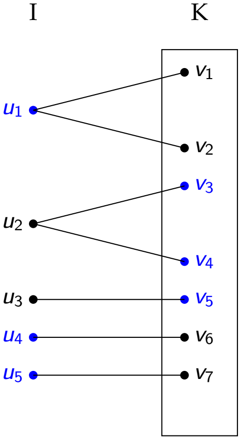
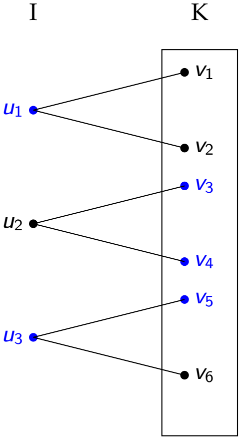


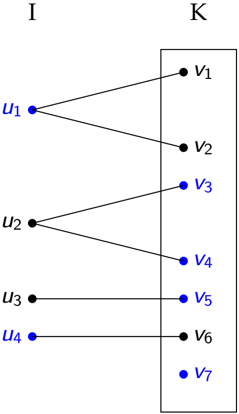
$S = \{u_1, u_4, u_5, v_3, v_4, v_5\}$

Example 2



$S = \{u_1, u_3, v_3, v_4, v_5\}$

Example 3



$S = \{u_1, u_4, v_3, v_4, v_5, v_7\}$

Steiner Path (G,R)

Instance: $K_{1,3}$-free split graph $G(K \cup I, E)$, Terminal set $R \subseteq V(G)$
Question: Find a minimum cardinality set $S \subseteq V(G) \setminus R$ such that $G[S \cup R]$ is a path ?

Observation

Given a claw free split graph $G(K \cup I, E)$ and R, if $|P \cap R| \geq 3$ then Steiner path does not exists, where $P = \{v \mid v \in V(G) \text{ and } d_G(v) = 1\}$

Theorem

Let $G(K \cup I, E)$ be a claw free split graph with $|I| \geq 4$ and $|P \cap R| \leq 2$. Then Steiner path always exists.

Observation

Given a claw free split graph $G(K \cup I, E)$ and R, if $|P \cap R| \geq 3$ then Steiner path does not exists, where $P = \{v \mid v \in V(G) \text{ and } d_G(v) = 1\}$

Theorem

Let $G(K \cup I, E)$ be a claw free split graph with $|I| \geq 4$ and $|P \cap R| \leq 2$. Then Steiner path always exists.

Let $W_0 = \{v \mid v \in I \cap R \text{ and } |N_G(v) \cap R| = 0\}$, $W_1 = \{v \mid v \in I \cap R \text{ and } |N_G(v) \cap R| = 1\}$.
Case-1: $|W_0| \geq 2$ and $|W_1| \geq 1$.
Case-2: $|W_0| \geq 1$ and $|W_1| \geq 2$.
In both the cases, minimum Steiner set is $2(|W_0| - 1) + |W_1|$.

$K_{1,4}$-free split graphs: MTOCD

$K_{1,4}$-free split graph G

- $\Delta^I = 2$
  - Labeled Graph L
  - Maximum Matching(M) in L
- $\Delta^I = 3$
  - Construct graph $G_2(\Delta^I = 2)$
  - Use solution from G_2, $|D| = |D_2| + i, 0 \leq i \leq 4$

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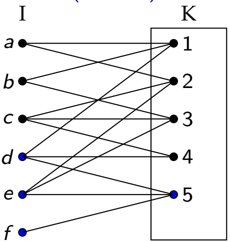
$G(\Delta^I = 3)$



$K_{1,4}$-free split graphs: MTOCD

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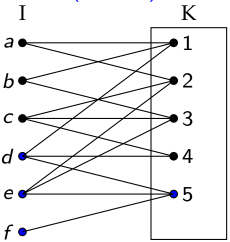
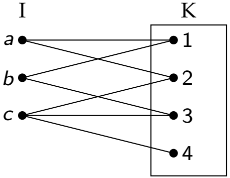


$G_2(\Delta^I = 2)$



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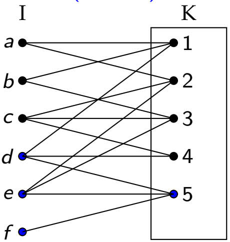
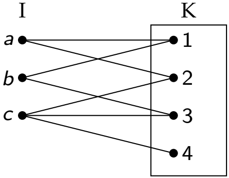
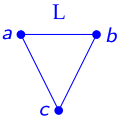


$G_2(\Delta^I = 2)$





$K_{1,4}$-free split graph G

- $\Delta^I = 2$
  - Restricted bipartite graph H
  - Collection of maximal paths($I - I$,$I - K$,$K - K$)
- $\Delta^I = 3$
  - Construct graph $G_2(\Delta^I = 2)$
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$K_{1,4}$-free split graphs: Longest Path

$K_{1,4}$-free split graph G

- $\Delta^I = 2$
  - Restricted bipartite graph H
  - Collection of maximal paths($I - I$,$I - K$,$K - K$)
- $\Delta^I = 3$
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  - Use solution from G_2

⋆ Longest Path in $K_{1,4}$-free split graphs is polynomial-time solvable.

⋆ Since the Hamiltonian path problem in $K_{1,5}$-free split graphs is NP-complete, Longest path problem in $K_{1,5}$-free split graphs is also NP-complete.

$K_{1,5}$-free split graphs: TOCDD is NP-Complete

TOCDD (G,k)

Instance: Graph $G(V,E)$, Integer $k \geq 0$
Question: Does there exist a set $D \subseteq V(G)$ such that $|D| \leq$ k and every vertex in G is adjacent to a vertex in D and G[V\D] is connected?

Exact 3 Cover(X,$\mathbb{C}$)

Instance: $X = \{x_1, x_2, \ldots, x_{3q}\}, \mathbb{C} = \{c_i \mid c_i \subset X, |c_i| = 3\}$.
Question: Does there exist $\mathbb{C}' \subseteq \mathbb{C}$ such that $\mathbb{C}'$ partitions X?

Exact 3 cover $\leq_P$ TOCDD

- Reduced graph - G
- $D = \{c_i \mid c_i \in \mathbb{C}^{'}\} \cup W \cup Z$
- $Y = \{y_1, y_2, ..., y_{3q}\}$
- $Z = \{z_1, z_2, ..., z_{3q}\}$
- $W = \{w_1, w_2, ..., w_{3q}\}$

TOCDD (G,k)

Instance: Graph $G(V,E)$, Integer $k \geq 0$
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Exact 3 Cover(X,ℂ)

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Exact 3 cover $\leq_P$ TOCDD

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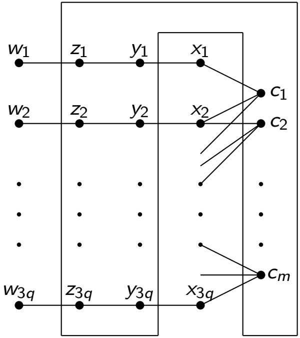
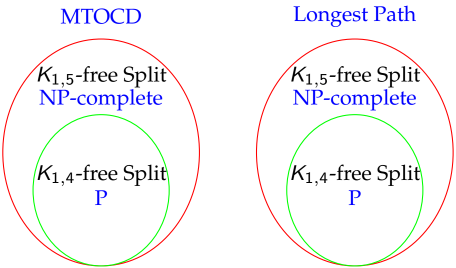
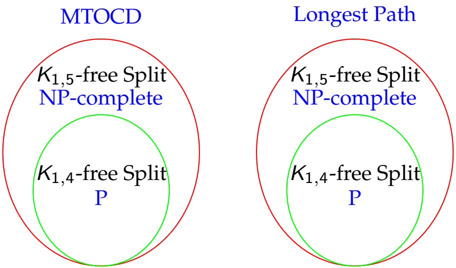
- ⋆ $W = \{w_1, w_2, \ldots, w_{3q}\}$

Reduction instances

- ⋆ *$K_{1,5}$-free Split Graphs*




MTOCD
$K_{1,5}$-free Split
NP-complete
$K_{1,4}$-free Split
P
Longest Path
$K_{1,5}$-free Split
NP-complete
$K_{1,4}$-free Split
P



- ⋆ Explore the Maximum-Cut problem in $K_{1,r}$-free split graphs, where $r \geq 4$.
- ⋆ Explore the Steiner path problem in $K_{1,r}$-free split graphs, where $r \geq 4$.

Thank You